Option Pricing Algorithms under Transaction Costs

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Abstract

Option replication and super-replication are discussed in a binomial tree setting, in the presence of proportional transaction costs. Pricing algorithms are associated with iterative procedures which return hedging strategies for the writer and for the buyer of the option.

Contents

Introduction

Under perfect market assumptions, the fair price of an option is derived from its replicating portfolio. Yet this method implies that adjusting the positions in the stocks and in the bonds holdings are costless, which is not the case when there are transactions costs. As a consequence, there can be alternative strategies which cost less but produce payoffs at least as valuable as the replicating strategy. That is, the replicating argument does not hold anymore when there are transactions costs. We will then introduce the notion of super-replication. In the first chapter, we will define all specifications of the market model and all preliminary notions. In the second one, we will give conditions for the absence of arbitrage under transactions costs. The third chapter will develop an algorithm for replication, and the last one will describe an algorithm for super-replication.

1. The Market Model

1.1 Introduction

In our market model, time runs in one-lengthed steps, that is, $t = 0, ..., T$, where T is the maturity time of the option. We will work in a binomial tree model, that is, a model in which the upward and the downward price movements follow a binomial distribution.

The following diagram represents a two-step binomial tree representing the stock price.

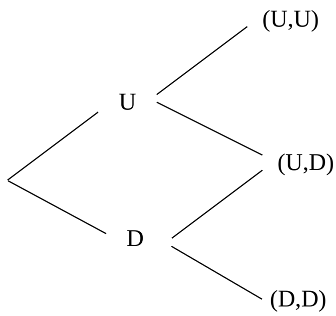


Figure 1.1: Two-step binomial tree

We will denote as u and d any upward and downward movement of the stock price. Hence, one characterization of the binomial framework is that the stock price in the (u, d) and the (d, u) states is the same.

We will also consider the finite probability space $(\Omega, Q, \mathcal{F})$, where $\mathcal{F}$ is the the collection of all subsets of Ω. In other words, for any $\omega \in \mathcal{F}$, we have $Q(\{\omega\}) < \infty$. Moreover, the measurable space (Ω, Q) is equipped with a filtration $\mathcal{F} = (\mathcal{F}_t)_t$ such that

$$\{\Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_T = \mathcal{F}.$$

For $t = 0, ..., T$, the sigma-field $\mathcal{F}_t$ defines the information available at time t. Hence, the filtration describes the evolution of the set of measurable events, as the information from the present becomes more and more precise.

Besides, we assume for simplicity that $Q(\{\omega\}) > 0$ for all $\omega \in \Omega$.

Notation 1.1.1. For $t = 0, ..., T$, an atom A of $\mathcal{F}_t$ is a subset of Ω such that $A \in \mathcal{F}_t$, and for any strict non-empty subset μ of A, we have $\mu \notin \mathcal{F}_t$. Let Ω_t be the collection of atoms of $\mathcal{F}_t$.

If we compare the filtration $\mathcal{F}$ to an event tree, the atoms of $\mathcal{F}_t$ correspond to time-t nodes of the tree, for $t = 0, \ldots, T$. At time 0, there is only one node Ω_0 which consists of the certain event Ω. At time T, Ω_T is the collection of single-element subsets of Ω which are identified with all scenarios $\omega \in \Omega$.

Example 1.1.2. In Figure 1.1, we have:

$$\Omega_1 = \{u, d\} \text{ where } u = \{uu, ud\} \text{ and } d = \{du, dd\};$$
$$\Omega_2 = \{uu, ud, du, dd\}.$$

Definition 1.1.3. Let $\mu \in \Omega_t$ and $\upsilon \in \Omega_{t+1}$. The node υ is a *successor node* of μ if $\upsilon \subset \mu$. We will denote as $succ\ \mu$ the set of successor nodes of μ.

The model contains two assets, namely a risk-free asset (bonds, or a bank account) and a risky asset (called a stock). Proportional transaction costs are applicable to the stock, in that at each time $t = 0, ..., T$, the trader can rebalance his positions in the stock and in the bond. A portfolio will be denoted as (α_t, β_t), where α_t (respectively β_t) represents the stock (respectively the bond) holding between time $t-1$ and time t. A sequence of portfolios (α_t, β_t), where $t = 0, ..., T+1$, form a *trading strategy* (or *hedging strategy*).

Notation 1.1.4. A trading strategy will be denoted as (α, β) or $(\alpha_t, \beta_t)_t$, where $0 \leq t \leq T+1$.

To re-balance his positions between each trading time $t = 0, \ldots, T$, we suppose that the trader can buy, sell and hold any amount of shares, whether integer or fractional, and can take any long or short position in the bond.

Because of the transaction costs, buying or selling a share does not necessarily mean paying or receiving the same amount of cash. In other words, the presence of transaction costs implies that in order to buy some shares, an investor will pay more than the price of the shares, whereas if he sells some shares, he will receive less amount of cash than the actual price of the shares. At a given time t, for $t = 0, ..., T$, a share can be respectively bought or sold at the *ask* or the *bid price*.

For $t = 0, ..., T$, let us denote by S_t^a (respectively S_t^b) the ask (respectively the bid) price. We naturally consider that $0 < S_t^b \leq S_t^a$ and that S_t^a and S_t^b are known at each time t, for $t = 0, \ldots, T$. For convenience, we assume that the interest rate paid by the bond is equal to 0 and that the bond price is equal to 1 for $t = 0, \ldots, T$.

For $t = 0, ..., T$, the *liquidation value* of the strategy (α, β), denoted as $V_t(\alpha_t, \beta_t)$, is the amount of cash provided by the strategy (α, β) if the stock is liquidated at time t, that is:

$$V_t(\alpha_t, \beta_t) = \alpha_t - \beta_t^- S_t^a + \beta_t^+ S_t^b,$$

where $\zeta^+ = \zeta \cdot 1_{[0,+\infty[}$ and $\zeta^- = -\zeta \cdot 1_{]-\infty,0]}$.

Definition 1.1.5. A trading strategy (α, β) is said to be *self-financing* if:

1. $\beta_0 = 0$;

2. For $t = 1, \ldots, T+1$, we have (α_t, β_t) is $\mathcal{F}_{t-1}$-measurable, that is, the positions in the stock and in the bond between time $t-1$ and time t depend only on the information available up to time $t-1$;

3. for $t = 0, ..., T$, we have
$$V_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) \geq 0. \tag{1.1}$$

Remark 1.1.6. Condition 2 ensures that all market participants have access to the same information and that no investor can foresee the future evolution of the stock price.

Notation 1.1.7. Let Φ be the set of self-financing strategies.

Definition 1.1.8. A strategy $(\alpha, \beta) \in \Phi$ is said to be an *arbitrage opportunity* if:

1. $\alpha_0 \leq 0$;
2. $V_T(\alpha_{T+1}, \beta_{T+1}) \geq 0$;
3. $\mathbb{E}_Q(V_T(\alpha_{T+1}, \beta_{T+1}) > 0) > 0$.

That is to say, an *arbitrageur* is a market participant who can make profit with no initial investment.

Our market model admits no arbitrage opportunities. In Chapter 2, we will give a necessary and sufficient condition for lack of arbitrage in the presence of transaction costs.

1.2 Replication and super-replication

1.2.1 European Options

An *call option* is an agreement which gives its owner, that is the *buyer of the option*, the right to buy a share at specified date $\hat{t}$ and at a fixed price K, called the *strike price* of the option. If he decides to exercise the option, then the *seller* will have to trade as previously agreed.

Let (ξ, ζ) be the payoff of an European option.

Example 1.2.1. A European call option with *physical delivery* is a European option which has the following payoff at maturity:

$$(\xi, \zeta) = \begin{cases} (-K, 1) & \text{if the option is exercised,} \\ (0, 0) & \text{else.} \end{cases}$$

In other words, an option with physical delivery allows its owner to actually receive one unit of the stock if he decides to exercise the call option.

A European call option with *cash settlement* is a European option such that

$$(\xi, \zeta) = ((S_T - K)^+, 0).$$

In other words, this kind of option involves no trading in stocks, whether its owner decides to exercise it or not.

Because of transactions costs, the payoffs of options with cash settlement and with physical delivery do not provide the same liquidation values at maturity time. Depending on the positions of the trader at the delivery time T, he will choose the option with physical delivery or the one with cash settlement.

Example 1.2.2. First, consider the case where the owner of a call option has no stock and no cash at the delivery time. In this case, the liquidation value of the option is given by:

Option settlement	If not exercised	If exercised
Cash settlement	0	$S_T - K$
Physical delivery	0	$S_T^b - K$

Since $S_T - K \geq S_T^b - K$, the owner of the option will prefer the cash settlement.

Now, let us consider the case where the buyer of the call option has no bond holdings and one short position in the stock at the delivery time. If he chooses the option with physical delivery, he can use the stock from the delivery to settle his short position. Hence, the liquidation value of the option is given by:

Option settlement	If not exercised	If exercised
Cash settlement	$-S_T^a$	$-S_T^a + S_T - K$
Physical delivery	$-S_T^a$	$-K$

In the presence of proportional transaction costs, we have $-K \geq -S_T^a + S_T - K$. As a result, the option buyer will prefer the option with delivery.

From now on, we will consider a European call option with payoff (ξ, ζ) and exercise time T.

1.2.2 Replicating and super-replicating strategies

In the following definitions, we will consider a European option with payoff (ξ, ζ) and exercice time T.

Definition 1.2.3. A strategy $(\alpha, \beta) \in \Phi$ *super-replicates the option* (ξ, ζ) *for the seller* if:

$$V_T(\alpha_T - \xi, \beta_T - \zeta) \geq 0. \tag{1.2}$$

Definition 1.2.4. A strategy $(\alpha, \beta) \in \Phi$ *super-replicates the option* (ξ, ζ) *for the buyer* if:

$$V_T(\alpha_T + \xi, \beta_T + \zeta) \geq 0. \tag{1.3}$$

Definition 1.2.3 means that by following the self-financing strategy (α, β), the seller will be able to deliver the portfolio (ξ, ζ) without risk. Likewise, Definition 1.2.4 means that the buyer who follows the self-financing strategy (α, β) will be able to leave the portfolio $(-\xi, -\zeta)$ and to end up with a solvent portfolio at time T upon receiving the portfolio (ξ, ζ).

A strategy $(\alpha, \beta) \in \Phi$ is said to *replicate* the option for the buyer (respectively for the seller) if the inequalities (1.1) and (1.2) (respectively (1.3)) are transformed into equalities. Hence, in the case of replication, the trader perfectly matches his liability, whereas in the case of super-replication, he is allowed to dominate his liability.

Remark 1.2.5. *Importance of replication and super-replication:*

When there are no transaction costs, replication gives the fair price of an option. That is, the price of the option is computed as the initial value of the replicating (or hedging) strategy.

Under transaction costs, there exist conditions under which the replicating strategy is still the cheapest way to price an option [BLPS92]. However, this is not always the case. In fact, at time $t = 0$, there may be some portfolios which dominate the initial replicating one.

Example 1.2.6. Consider the following one-step binomial tree [RZ06] which represents the stock price, where the parenthesis represents the payoff of the call option.

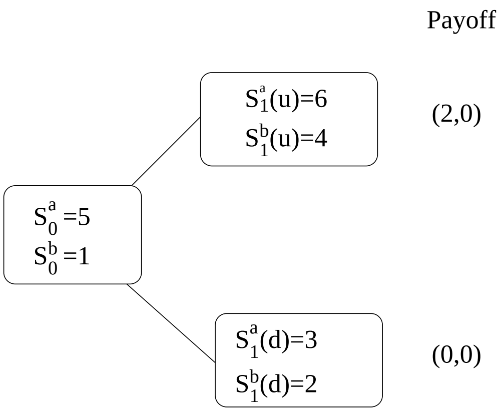


Figure 1.2: Event tree representing the bid-ask spread of the stock price at each time

Let (α, β) be the replicating strategy of this option.

Following the definition of a perfect hedging strategy, we have:

$$(\alpha_2(u), \beta_2(u)) = (2, 0),$$
$$(\alpha_2(d), \beta_2(d)) = (0, 0).$$

If the inequality (1.2) is transformed into an equality, we get:

$$\alpha_1 + \beta_1^+ S_1^b(u) - \beta_1^- S_1^a(u) = \alpha_2(u) + \beta_2^+(u) S_1^b(u) - \beta_2^-(u) S_1^a(u),$$
$$\alpha_1 + \beta_1^+ S_1^b(d) - \beta_1^- S_1^a(d) = \alpha_2(d) + \beta_2^+(d) S_1^b(d) - \beta_2^-(d) S_1^a(d),$$

that is,

$$\alpha_1 + 4\beta_1^+ - 6\beta_1^- = 2, \tag{1.4}$$
$$\alpha_1 + 2\beta_1^+ - 3\beta_1^- = 0. \tag{1.5}$$

To solve these equations, let us first suppose that $\beta_1 \geq 0$. We then have:

$$\alpha_1 + 4\beta_1 = 2,$$
$$\alpha_1 + 2\beta_1 = 0,$$

which gives $(\alpha_1, \beta_1) = (-2, 1)$.

Now, let us assume that $\beta_1 \leq 0$. Thus:

$$\alpha_1 + 6\beta_1 = 2, \tag{1.6}$$
$$\alpha_1 + 3\beta_1 = 0. \tag{1.7}$$

Subtracting equation (1.6) from (1.7), we get $3\beta_1 = 2$, that is, $\beta_1 = 2/3$, which is in contradiction with the assumption $\beta_1 \leq 0$.

Therefore, the initial replicating portfolio is necessarily given by

$$(\alpha_1, \beta_1) = (-2, 1).$$

It is straightforward to check that, in this case, equations (1.4) and (1.5) actually hold.

The initial value of the perfect hedging strategy is thus given by:

$$\alpha_0 = V_0(\alpha_1, \beta_1) = V_0(-2, 1) = -2 + 1 \times 5 = 3.$$

However, if this value is used to price the option, then there would be an arbitrage opportunity. Indeed, one could sell the option at 3, buy two units of bonds and keep a strategy (α, β) such that $(\alpha_1, \beta_1) = (\alpha_2, \beta_2) = (2, 0)$: If the up-state occurs, he would make an arbitrage profit of 1; if the $down$-state occurs, he would make an arbitrage profit of 3.

Hence, in the presence of transaction costs, it is possible to find strategies which cost less than the replicating one, but which produce payoffs more valuable than the latter. As a result, the replicating argument does not always holds. This is the reason why the notion of super-replication was introduced in modern finance.

The basic principle of super-replication consists in the introduction of weaker budget constraints at each time-step $t = 0, \ldots, T$. In other words, for $t = 0, \ldots, T$, instead of having a portfolio (α_t, β_t) which value is exactly enough to hedge his positions, the trader may have more than actually needed. This weakness of budget constraints is expressed by the inequality " $\geq$" in Definitions 1.2.3 and 1.2.4. This domination of the liabilities may lead to lower transaction costs, thereby to a smaller initial cost. Hence, as shown in Example 1.2.6, under transaction costs, a super-replicating strategy may be cheaper than a perfect hedging one.

Similarly to the case of the stock, the *ask price of an option* is the price at which the option can be bought on demand. Therefore, it is obtained by hedging the option writer's positions.

The *bid price of an option* is the price at which the option can be sold on demand. It is derived from the hedging strategy of the option buyer's positions.

Definition 1.2.7. The ask price (respectively the bid price) of the option is defined by:

$$\pi^a(\xi, \zeta) = \min\{\alpha_0 | (\alpha, \beta) \text{ super-replicates } (\xi, \zeta) \text{ for the seller}\}, \tag{1.8}$$
$$\pi^b(\xi, \zeta) = \max\{-\alpha_0 | (\alpha, \beta) \text{ super-replicates } (\xi, \zeta) \text{ for the buyer}\}. \tag{1.9}$$

Intuitively, the ask price can be interpreted as the minimum amount of money a trader would have to invest initially to get a profit at least as valuable as the option's payoff. In contrast, the bid price can be seen as the maximum amount of cash a buyer can borrow against the proceeds of the option, that is, using the option as a guarantee that he will repay his loan.

Observe that a replicating strategy is also a super-replicating strategy. It turns out that there exists cases [BLPS92] where the term "super-replicate" in Definition 1.2.7 can be replaced by "replicate". For instance, in the case of "small proportional transaction costs", a trader will choose strict replication.

Remark 1.2.8. Since we consider discrete-time models, the sets described in equations (1.8) and (1.9) are closed and respectively bounded below and above. Hence, the maximum and the minimum are attained.

Remark 1.2.9. Let (α^a, β^a) (respectively (α^b, β^b)) $\in \Phi$ be any strategy super-replicating (ξ, ζ) for the seller (respectively (ξ, ζ) for the buyer), such that $\pi^a(\xi, \zeta) = \alpha_0^a$ (respectively $\pi^b(\xi, \zeta) = \alpha_0^b$). We will show that (α^a, β^a) (respectively (α^b, β^b)) is effectively an optimal hedging strategy for the seller (respectively the buyer) of the option.

The following transformation between the long and the short position provides a way to deduce the bid price and the super-hedging strategy of the option buyer.

$$\begin{aligned}
\pi^b(\xi, \zeta) &= \max\{-\alpha_0 | (\alpha, \beta) \text{ super-replicates } (\xi, \zeta) \text{ for the buyer}\} \\
&= \max\{-\alpha_0 | (\alpha, \beta) \text{ such that } V_T(\alpha_T + \xi, \beta_T + \zeta) \geq 0\} \\
&= -\min\{\alpha_0 | (\alpha, \beta) \text{ super-replicates } (-\xi, -\zeta) \text{ for the seller}\}. \\
\pi^b(\xi, \zeta) &= -\pi^a(-\xi, -\zeta).
\end{aligned}$$

Therefore, it is enough to give an algorithm to compute the ask price and to hedge the long call position.

2. Lack of Arbitrage and Equivalent Martingale Measures

In finance theory, it is well-known that the absence of arbitrage is equivalent to the existence of a risk-neutral probability measure when trading in frictionless discrete-time market models [CZ03]. In this section, we will prove that this holds under proportional transaction costs.

2.1 Convexity

Let $A \subset \mathbb{R}$ be a set.

Definition 2.1.1. Any set A is a *convex* if:

$$\lambda a_1 + (1 - \lambda) a_2 \in A,$$

for any $a_1, a_2 \in A$, for any $\lambda \in (0, 1)$.

The set A is a *cone* if for any $a \in A$, for any $\lambda \geq 0$, $\lambda a \in A$.

We will use the following theorem [EK98].

Separating Hyperplane Theorem. Let C a closed convex cone in $\mathbb{R}^n$ and K a compact convex set. If $C \bigcap K = \emptyset$, then there exists a linear functional $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $F(c) < 0$ for any $c \in C$ and $F(k) \geq 0$ for any $k \in K$.

The following result will play an important role in our further deliberations.

Lemma 2.1.2. *The set Φ is a closed convex cone.*

Proof. See Appendix for details. □

Let χ be the vector space of all random variables $X : \Omega \longrightarrow \mathbb{R}$.

Notation 2.1.3. Let χ^+ the subspace of χ consisting of positive random variables on Ω. In other words:

$$\chi^+ = \{X \in \chi : X \geq 0\}.$$

Let $\mathcal{M}$ the subset of χ defined as:

$$\mathcal{M} = \{X \in \chi | \exists (\alpha, \beta) \in \Phi : \alpha_0 \leq 0, V_T(\alpha_{T+1}, \beta_{T+1}) \geq X\}.$$

In fact, some $X \in \mathcal{M} - \{0\}$ can be considered as the final liquidation value of a given arbitrage opportunity (α, β).

Lemma 2.1.4. *The set $\mathcal{M}$ is a closed convex cone.*

Proof. See Appendix for details. □

2.2 Equivalent Martingale Measure

Definition 2.2.1. A probability measure P on Ω is *equivalent* to Q if and only if:

$$P(\omega) = 0 \Leftrightarrow Q(\omega) = 0, \text{ for all } \omega \in \Omega.$$

Definition 2.2.2. A probability measure $P : \Omega \longrightarrow (0, 1)$ is a *risk-neutral probability* if:

1. P is equivalent to Q;
2. There exists a martingale S under P such that for $t = 0, ..., T$, we have:

$$S_t^b \leq S_t \leq S_t^a.$$

Notation 2.2.3. Let us denote as $\mathcal{P}$ the set of (P, S) such that P is a risk-neutral probability and S a martingale under P satisfying $S_t^b \leq S_t \leq S_t^a$, for $t = 0, ..., T$. In other words, any $(P, S) \in \mathcal{P}$ satisfies Conditions 1 and 2.
By extension, let us denote as $\bar{\mathcal{P}}$ the set of pairs (P, S) such that P is a probability measure on Ω and any $(P, S) \in \bar{\mathcal{P}}$ obeys to Condition 2.

Lemma 2.2.4. *For any $(\alpha, \beta) \in \Phi$, the adapted process $\alpha + \beta S$ is a supermartingale, i.e*

$$\mathbb{E}_P(\alpha_{t+1} + \beta_{t+1} S_{t+1} | \mathcal{F}_t) \leq \alpha_t + \beta_t S_t,$$

for $t = 0, \ldots, T$.

Proof. The inequality $S_t^b \leq S_t \leq S_t^a$ implies that:

$$-(\beta_t - \beta_{t+1})^- S_t^a + (\beta_t - \beta_{t+1})^+ S_t^b \leq (\beta_t - \beta_{t+1}) S_t, \tag{2.1}$$

for $t = 0, ..., T$. Besides, since $(\alpha, \beta) \in \Phi$, we have

$$\mathbb{E}_P(V_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1})) \geq 0.$$

It follows that:

$$\alpha_{t+1} - \alpha_t \leq -(\beta_t - \beta_{t+1})^- S_t^a + (\beta_t - \beta_{t+1})^+ S_t^b. \tag{2.2}$$

Equations (2.1) and (2.2) give:

$$\alpha_{t+1} - \alpha_t \leq (\beta_t - \beta_{t+1}) S_t. \tag{2.3}$$

The fact that S is a martingale under P together with equation (2.3) imply that:

$$\mathbb{E}_P(\alpha_{t+1} + \beta_{t+1} S_{t+1} | \mathcal{F}_t) = \alpha_{t+1} + \beta_{t+1} S_t \leq \alpha_t + \beta_t S_t,$$

for $t = 0, ..., T$, which means that $\alpha + \beta S$ is a supermartingale under P. ☐

Theorem 2.2.5. *[Tok04] The market admits no arbitrage opportunities if and only if there exists a risk-neutral probability P.*

Proof. First, let us suppose that the market admits a risk-neutral probability P. Let S be a martingale under P such that for any t,

$$S_t^b \leq S_t \leq S_t^a. \tag{2.4}$$

Suppose that there exists a strategy $(\hat{\alpha}, \hat{\beta}) \in \Phi$ such that

$$\hat{\alpha}_0 \leq 0, \tag{2.5}$$

$$V_T(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1}) \geq 0. \tag{2.6}$$

Let us prove that we necessarily have $\mathbb{E}_Q(V_T(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1})) = 0$.

Since $S_T^b \leq S_T \leq S_T^a$, we have:

$$V_T(\alpha_{T+1}, \beta_{T+1}) = \alpha_{T+1} - \beta_{T+1}^- S_T^a + \beta_{T+1}^+ S_T^b \leq \alpha_{T+1} + \beta_{T+1} S_T \leq \alpha_T + \beta_T S_T. \tag{2.7}$$

According to Lemma 2.2.4, $\alpha + \beta S$ is a supermartingale. Hence, equation (2.7) implies:

$$\mathbb{E}_P(V_T(\alpha_{T+1}, \beta_{T+1})) \leq \mathbb{E}_P(\alpha_T + \beta_T S_T) \leq \alpha_0. \tag{2.8}$$

Now, let us consider the strategy $(\hat{\alpha}, \hat{\beta})$ defined in (2.5) and (2.6).

According to (2.6), (2.8) and (2.5), we have:

$$0 \leq \mathbb{E}_P(V_T(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1})) \leq \hat{\alpha}_0 \leq 0.$$

Thus, $\mathbb{E}_P(V_T(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1})) = 0$. Since P is equivalent to Q, this implies that

$$\mathbb{E}_Q(V_T(\hat{\alpha}_{T+1}, \hat{\beta}_{T+1})) = 0,$$

and the third condition of arbitrage is violated. It follows that there are no arbitrage opportunities.

Conversely, let us suppose that there are no arbitrage opportunities. Using Notation 2.1.3, one can easily see that χ^+ is a convex cone. Besides, we have seen in Lemma 2.1.4 that $\mathcal{M}$ is a convex cone.

Since there are no arbitrage opportunities, $\chi^+ \bigcap \mathcal{M} = \{0\}$. Indeed, let us suppose that there exists $X \in \chi^+ \bigcap \mathcal{M} - \{0\}$. Hence, there would be a self-financing strategy (α, β) such that:

$$\alpha_0 \leq 0. \tag{2.9}$$

and

$$V_T(\alpha_{T+1}, \beta_{T+1}) \geq X.$$

The definition of χ^+ together with the fact that $X \neq 0$ imply that there would exist at least one $\omega \in \Omega$ such that:

$$V_T(\alpha_{T+1}(\omega), \beta_{T+1}(\omega)) \geq X(\omega) > 0, \tag{2.10}$$

and equations (2.9) and (2.10) would amount to arbitrage.

Let $K \subset \chi^+$ be the compact convex set defined as

$$K = \left\{ X \in \chi^+ \text{ such that } \sum_{\omega \in \Omega} X(\omega) = 1 \right\}.$$

Therefore, $0 \notin K$. Since $\chi^+ \bigcap \mathcal{M} = \{0\}$, we have $K \bigcap \mathcal{M} = \emptyset$, and the Separating Hyperplane Theorem implies that there exists a vector $(z(\omega), \omega \in \Omega)$ such that:

1. $\sum_{\omega\in\Omega} z(\omega)X(\omega) > 0$, for any $X \in K$;
2. $\sum_{\omega\in\Omega} z(\omega)X(\omega) \leq 0$, for any $X \in \mathcal{M}$.

Fix $\tilde{\omega} \in \Omega$. Let us consider $\tilde{X} : \Omega \longrightarrow \mathbb{R}$ defined as $\tilde{X}(\omega) = 1_{\{\tilde{\omega}\}}$, where 1_A is the indicator function of the set A.

We thus have $\tilde{X} \in K$ and Condition 1 implies that:

$$z(\tilde{\omega}) = \sum_{\omega\in\Omega} z(\omega)\tilde{X}(\omega) > 0.$$

Since $\tilde{\omega}$ is arbitrary, we necessarily have $z(\omega) > 0$ for any $\omega \in \Omega$. Therefore:

$$\sum_{\omega'\in\Omega} z(\omega') > 0. \tag{2.11}$$

This implies that P defined as:

$$P(\omega) = \frac{z(\omega)}{\sum_{\omega'\in\Omega} z(\omega')}$$

for $\omega \in \Omega$ is a probability measure on Ω. Besides, P is equivalent to Q.

Condition 2 and equation (2.11) imply that, for any $X \in \mathcal{M}$:

$$\mathbb{E}_P(X) = \sum_{\omega\in\Omega} P(\omega)X(\omega) = \frac{1}{\sum_{\omega'\in\Omega} z(\omega')} \sum_{\omega\in\Omega} z(\omega)X(\omega) \leq 0. \tag{2.12}$$

Let us now prove the existence of a martingale $S = (S_t)_{0\leq t\leq T}$ under P such that $S^b_t \leq S_t \leq S^a_t$, for any $t = 0, \dots, T$.

For $t = 0, ..., T-1$, let ϕ^a_t (respectively ϕ^b_t) be the set of strategies defined as:

$$\begin{aligned}
\phi^a_t &= \{(\alpha,\beta) \in \Phi |\text{ for all } m \leq t, (\alpha_m,\beta_m) = (0,0), \alpha_{t+1} = -\beta^-_{t+1}S^a_{t+1} + \beta^+_{t+1}S^b_{t+1}, \beta_{T+1} = 1\},\\
\phi^b_t &= \{(\alpha,\beta) \in \Phi |\text{ for all } m \leq t, (\alpha_m,\beta_m) = (0,0), \alpha_{t+1} = -\beta^-_{t+1}S^a_{t} + \beta^+_{t+1}S^b_{t+1}, \beta_{T+1} = -1\}.
\end{aligned}$$

In other words, ϕ^a_t is the set of self-financing strategies consisting in no trading before time $t+1$ and ending up with a long position in the stock. Likewise, ϕ^b_t is the set of self-financing strategies consisting in no trading before time $t+1$ and ending up with a short position in the stock.

Note ϕ^a_t and ϕ^b_t are subsets of Φ determined by equalities. Using a similar argument as in Lemma 2.1.2, one can see that ϕ^a_t and ϕ^b_t are closed convex cones.

Let (α^a, β^a) be the strategy consisting in no trading in the stock and in the bond before time T and buying one unit of the stock at time T. Therefore, (α^a, β^a) is uniquely determined by its portfolio $(\alpha^a_{T+1}, \beta^a_{T+1}) = (-S^a_T, 1)$. In a similar way, let (α^b, β^b) be the strategy consisting in no trading in the stock and in the bond before time T and selling one unit of the stock at time T.

That is to say, (α^b, β^b) is determined by its final portfolio $(\alpha^b_{T+1}, \beta^b_{T+1}) = (S^b_T, -1)$.

Note that the strategy (α^a, β^a) (respectively (α^b, β^b)) is the only element in ϕ^a_{T-1} (respectively ϕ^b_{T-1}).

Let us define $\hat{S}^a_t$ and $\hat{S}^b_t$ as follows:

- for $t = 0, ..., T-1$,

$$\hat{S}^a_t = \min_{(\alpha,\beta)\in\phi^a_t} \mathbb{E}(-\alpha_{T+1}|\mathcal{F}_t),$$
$$\hat{S}^b_t = \max_{(\alpha,\beta)\in\phi^b_t} \mathbb{E}(\alpha_{T+1}|\mathcal{F}_t).$$

- for $t = T$,

$$\hat{S}^a_T = S^a_T \text{ and } \hat{S}^b_T = S^b_T.$$

Let us show that for any $t = 0, ..., T$, we have:

$$S^b_t \leq \hat{S}^b_t \leq \hat{S}^a_t \leq S^a_t. \tag{2.13}$$

For $t = T$, the result is straightforward.

For $t \leq T-1$, let (α^m, β^m) be the strategy consisting in buying a stock at time t and keeping this stock until the maturity time. We then have $(\alpha^m_{T+1}, \beta^m_{T+1}) = (-S^a_t, 1)$. Similarly, let (α'^m, β'^m) be the strategy consisting in selling a stock at time t and keeping this short position until the maturity time, that is, $(\alpha'^m_{T+1}, \beta'^m_{T+1}) = (S^b_t, -1)$.

We then have $(\alpha^m, \beta^m) \in \phi^a_t$ and $(\alpha'^m, \beta'^m) \in \phi^b_t$. It follows that $\hat{S}^a_t \leq S^a_t$ and $S^b_t \leq \hat{S}^b_t$.

Let (α, β) and (α', β') be any strategies in ϕ^a_t and ϕ^b_t such that

$$\hat{S}^a_t = \mathbb{E}(-\alpha_{T+1}|\mathcal{F}_t),$$
$$\hat{S}^b_t = \mathbb{E}({\alpha'}_{T+1}|\mathcal{F}_t).$$

Let B be any time-t node, where $t \leq T-1$. We denote as $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\alpha}', \hat{\beta}')$ the strategies composed by "empty" portfolios except on the node B. In other words, we have

$$(\hat{\alpha}(\mu), \hat{\beta}(\mu)) = (0, 0),$$
$$(\hat{\alpha}'(\mu), \hat{\beta}'(\mu)) = (0, 0),$$

for any μ such that $B \notin succ\ \mu$.

One can see that $(\hat{\alpha}, \hat{\beta})$ and $(\hat{\alpha}', \hat{\beta}')$ are self-financing.

Besides, since $(\alpha, \beta) \in \phi^a_t$ and $(\alpha', \beta') \in \phi^b_t$,

$$\hat{\beta}_{T+1} + \hat{\beta}'_{T+1} = 1 - 1 = (1-1)1_B = 0.$$

Lemma 2.1.2 implies that $(\hat{\alpha}+\hat{\alpha}', \hat{\beta}+\hat{\beta}') \in \Phi$. Therefore,

$$V_T(\hat{\alpha}_{T+1}+\hat{\alpha}'_{T+1}, \hat{\beta}_{T+1}+\hat{\beta}'_{T+1}) \in \mathcal{M}.$$

As a result, after equation (2.12), it follows that:

$$\mathbb{E}_P(\hat{\alpha}_{T+1}+\hat{\alpha}'_{T+1}) = \mathbb{E}_P(V_T(\hat{\alpha}_{T+1}+\hat{\alpha}'_{T+1}, \hat{\beta}_{T+1}+\hat{\beta}'_{T+1})) \leq 0,$$

Hence,

$$\mathbb{E}_P(\hat{\alpha}'_{T+1}) \leq \mathbb{E}_P(-\hat{\alpha}_{T+1}),$$

which implies $\hat{S}^b_t \leq \hat{S}^a_t$ on the time-t node B.

Since B is arbitrary, we have $S^b_t \leq S^a_t$ for any time-t node, where $t = 0, ..., T-1$.

Now, let us prove that $\hat{S}^a$ is a submartingale under P, *i.e*

$$\mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t) \geq \hat{S}^a_t.$$

For $t = 0, ..., T-1$, we have

$$\mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_\mathrm{t}) = \mathbb{E}_P[\min_{(\alpha,\beta)\in\phi^a_{t+1}} \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_{t\,+\,1})|\mathcal{F}_t].$$

Since $\phi^a_{t+1} \subset \phi^a_t$,

$$\mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t) \geq \mathbb{E}_P[\min_{(\alpha,\beta)\in\phi^a_t} \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_{t+1})|\mathcal{F}_t]. \tag{2.14}$$

Let $(\alpha^{min}, \beta^{min}) \in \phi^a_t$ such that $\min_{(\alpha,\beta)\in\phi^a_t} \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_{t\,+\,1}) = \mathbb{E}_P(-\alpha^{min}_{T+1}|\mathcal{F}_{t\,+\,1})$.

The tower property together with the fact that $(\alpha^{min}, \beta^{min})$ belongs to ϕ^a_t give:

$$\begin{aligned}
\mathbb{E}_P[\min_{(\alpha,\beta)\in\phi^a_t} \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_{t\,+\,1})|\mathcal{F}_t] &= \mathbb{E}_P[\mathbb{E}_P(-\alpha^{min}_{T+1}|\mathcal{F}_{t\,+\,1})|\mathcal{F}_t] \\
&= \mathbb{E}_P(-\alpha^{min}_{T+1}|\mathcal{F}_t) \\
&\geq \min_{(\alpha,\beta)\in\phi^a_t} \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_t).
\end{aligned} \tag{2.15}$$

Equations (2.14) and (2.15) imply that

$$\mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t) \geq \hat{S}^a_t. \tag{2.16}$$

For $t = T$, we have:

$$\mathbb{E}_P(\hat{S}^a_T|\mathcal{F}_{T-1}) = \mathbb{E}_P(S^a_T|\mathcal{F}_{T-1}) = \mathbb{E}_P(-\alpha_{T+1}|\mathcal{F}_{T-1}) = \hat{S}^a_{T-1}.$$

Similarly, for $t = 0, ..., T-1$, we find that

$$\mathbb{E}_P(\hat{S}^b_{t+1}|\mathcal{F}_t) \leq \hat{S}^b_t \text{ and}$$
$$\mathbb{E}_P(\hat{S}^b_T|\mathcal{F}_{T-1}) = \hat{S}^b_{T-1}.$$

We then have shown that for $t = 0, ..., T$:

$$\mathbb{E}_P(\hat{S}^b_{t+1}|\mathcal{F}_t) \leq \hat{S}^b_t \leq \hat{S}^a_t \leq \mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t). \tag{2.17}$$

Now, let us construct a martingale $S = (S_t)_{0\leq t\leq T}$ such that for any $t = 0, \ldots, T$, we have $S^b_t \leq S_t \leq S^a_t$. Equation (2.13) implies that it is sufficient to construct S such that for each t,

$$\hat{S}^b_t \leq S_t \leq \hat{S}^a_t. \tag{2.18}$$

If condition (2.18) holds, then, according to equation (2.17):

$$\mathbb{E}_P(\hat{S}^b_{t+1}|\mathcal{F}_t) \leq S_t \leq \mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t).$$

As a result, there exists an adapted random variable λ such that:

$$\begin{aligned} S_t &= \lambda\mathbb{E}_P(\hat{S}^a_{t+1}|\mathcal{F}_t) + (1-\lambda)\mathbb{E}_P(\hat{S}^b_{t+1}|\mathcal{F}_t) \\ &= \mathbb{E}_P[(\lambda\hat{S}^a_{t+1} + (1-\lambda)\hat{S}^b_{t+1})|\mathcal{F}_t]. \end{aligned}$$

By taking $S_{t+1} = \lambda\hat{S}^a_{t+1} + (1-\lambda)\hat{S}^b_{t+1}$ for $t = 0, ..., T-1$, the process S is a martingale under P. $\square$

3. Boyle-Vorst Algorithm

Boyle and Vorst [BV92] extended the Cox-Ross Rubinstein [CRR79] binomial option pricing model to include proportional transaction costs.

Let $\hat{u}$ and $\hat{d}$ be the parameters of the Cox-Ross-Rubinstein (CRR) binomial model. Let S_t be the price of one unit of the stock at time-t on a given node on the binomial tree. At time $t+1$, S_t goes either up to $S_{t+1}(u) = S_t\hat{u}$ or down to $S_{t+1}(d) = S_t\hat{d}$. The basic structure of the CRR model is described in Figure 3.1.

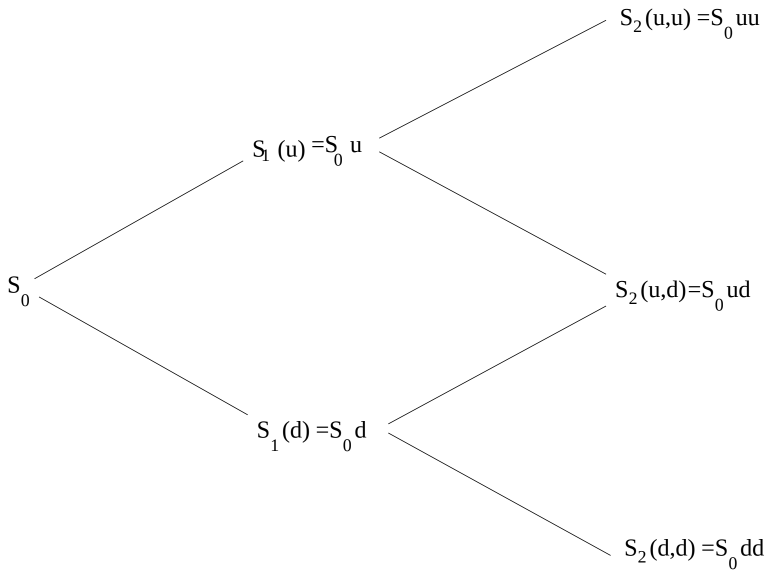


Figure 3.1: A two-step CRR binomial tree describing the stock price

In their model, Boyle and Vorst considered settlement with delivery where the terminal positions are in stocks (see Example 1.2.1). They assumed the transaction cost to be a fraction (k) of the amount of shares traded. In other words, at each time $t = 0, \ldots, T$, we have in fact:

$$S_t^a = S_t(1+k), \tag{3.1}$$
$$S_t^b = S_t(1-k). \tag{3.2}$$

In both cases of a long and a short call position, they used the no-arbitrage principle. However, they added two more restrictions to compute the option buyer's positions.

3.1 Replication of a Long Call Position

3.1.1 Basic formulae

A backward iteration procedure is used to compute the replicating portfolio of the call option. From the portfolios $(\alpha_{t+1}(u), \beta_{t+1}(u))$ and $(\alpha_{t+1}(d), \beta_{t+1}(d))$ held at time $t+1$, we deduce the portfolio (α_t, β_t) using the following equations:

$$\beta_t S_{t-1}\hat{u} + \alpha_t = \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(u) + k|\beta_t - \beta_{t+1}(u)| S_{t-1}\hat{u}, \tag{3.3}$$

$$\beta_t S_{t-1}\hat{d} + \alpha_t = \beta_{t+1}(d) S_{t-1}\hat{d} + \alpha_{t+1}(d) + k|\beta_t - \beta_{t+1}(d)| S_{t-1}\hat{d}. \tag{3.4}$$

Equation (3.3) (respectively (3.4)) means that the value of the replicating portfolio held at time t is exactly enough to pay the replicating portfolio held at time $t+1$ in the up-state (respectively the down-state), including the transactions costs.

3.1.2 Formal algorithm

Theorem 3.1.1. *There exists a unique solution (α_t, β_t) to equations (3.3) and (3.4) for each $t = 1, \ldots, T$. Moreover, we then have:*

$$\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u). \tag{3.5}$$

In particular, we get a unique initial portfolio (α_1, β_1).

Remark 3.1.2. Note that if (3.5) holds, we get:

$$(1+k)\beta_t S_{t-1}\hat{u} + \alpha_t = (1+k)\beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(u),$$
$$(1-k)\beta_t S_{t-1}\hat{d} + \alpha_t = (1-k)\beta_{t+1}(d) S_{t-1}\hat{d} + \alpha_{t+1}(d),$$

that is,

$$\alpha_t - \alpha_{t+1}(u) - (\beta_t - \beta_{t+1}(u))^- S_{t-1}\hat{u}(1+k) = 0, \tag{3.6}$$

$$\alpha_t - \alpha_{t+1}(d) + (\beta_t - \beta_{t+1}(d))^+ S_{t-1}\hat{d}(1-k) = 0, \tag{3.7}$$

Equation (3.6) means that, in the up-state, the additional amount of cash comes from selling $\beta_{t+1}(u) - \beta_t$ units of shares at the price $S_{t-1}\hat{u}(1+k) = S_t(u)(1+k)$. In a similar manner, equation (3.7) says that, in the down-state, the cash withdrawn is used to purchase $\beta_t - \beta_{t+1}(d)$ units of shares at the price $S_{t-1}\hat{d}(1-k) = S_t(d)(1-k)$. Observe that if we make the transformation

$$S_{t-1}\hat{u}(1+k) \mapsto S_t^a(u) \text{ and } S_{t-1}\hat{d}(1-k) \mapsto S_t^b(d),$$

we get:

$$\alpha_t - \alpha_{t+1}(u) - (\beta_t - \beta_{t+1}(u))^- S_t^a(u) = 0,$$
$$\alpha_t - \alpha_{t+1}(d) + (\beta_t - \beta_{t+1}(d))^+ S_t^b(d) = 0,$$

i.e,

$$V_t(\alpha_t - \alpha_{t+1}, \beta_t - \beta_{t+1}) = 0,$$

for $t = 0, \ldots, T$, which is consistent with the definition of a self-financing strategy given in Chapter 1.

Besides, equations (3.6) and (3.7) imply that the previous theorem transforms the non-linear system of equations to a linear one, and gives a straightforward method to compute the replicating portfolio of an option, including the adjustments implied by the transaction costs.

Boyle-Vorst's algorithm for the long call option works as follows:

- For $t = T + 1$, we put
$$\begin{cases} \alpha_{T+1} & = \xi, \\ \beta_{T+1} & = \zeta. \end{cases}$$
Hence, for $t = T$, $(\alpha_{t+1}, \beta_{t+1})$ is given.

- For $t = 1, \ldots, T$, we solve linear equations (3.6) and (3.7) for α_t and β_t. We iterate the process until time $t = 1$.

- For $t = 0$, we put
$$\begin{cases} \alpha_0 & = V_0(\alpha_1, \beta_1) = \alpha_1 - \beta_1^- S_0^a + \beta_1^+ S_0^b, \\ \beta_0 & = 0, \end{cases}$$
where S_0^a and S_0^b are given by equations (3.1) and (3.2).

Example 3.1.3. Let us use Theorem 3.1.1 to replicate a long call option, considering the binomial tree described in Figure 3.2. In our example, we consider $k = 10\%$.

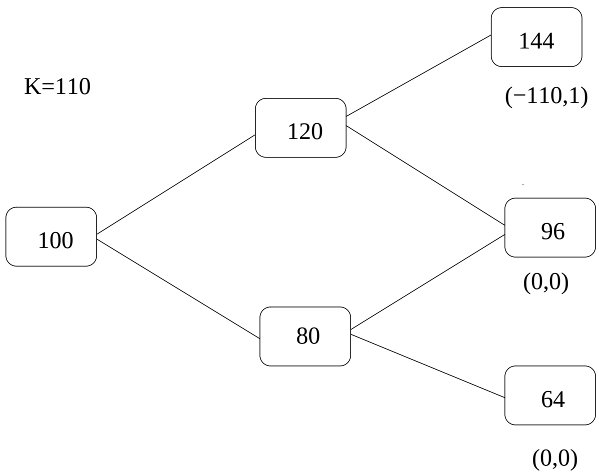


Figure 3.2: Example of a CRR binomial tree with $\hat{u} = 1.2$, $\hat{d} = 0.8$.

In order to minimize errors, we will use approximation results only for (α_1, β_1).

Using Remark 3.1.2, we get:

$$(1 + 10\%) \times 144 \times \beta_2(u) + \alpha_2(u) = (1 + 10\%) \times 1 \times 144 - 110, \tag{3.8}$$
$$(1 - 10\%) \times 96 \times \beta_2(u) + \alpha_2(u) = 0. \tag{3.9}$$

By subtracting equation (3.8) from (3.9), we get:

$$\beta_2(u) = \frac{(1 + 10\%) \times 144 - 110}{(1 + 10\%) \times 144 - (1 - 10\%) \times 96} = \frac{121}{180}.$$

Substituting in (3.9), we find that $\alpha_2(u) = -58.08$.

Similarly, we have:

$$(1 + 10\%) \times 96 \times \beta_2(d) + \alpha_2(d) = 0, \tag{3.10}$$

$$(1 - 10\%) \times 64 \times \beta_2(d) + \alpha_2(d) = 0. \tag{3.11}$$

Subtracting equation (3.10) from (3.11), we obtain $\beta_2(d) = 0$. Substituting in equation (3.11), we get $\alpha_2(d) = 0$.

At time $t = 0$, we have:

$$(1 + 10\%) \times 120 \times \beta_1 + \alpha_1 = (1 + 10\%) \times 120 \times \frac{121}{180} - 58.08, \tag{3.12}$$

$$(1 - 10\%) \times 80 \times \beta_1 + \alpha_1 = 0. \tag{3.13}$$

We find

$$\beta_1 = \frac{(1 + 10\%) \times 120 \times \frac{121}{180} - 58.08}{(1 + 10\%) \times 120 - (1 - 10\%) \times 80} = \frac{2299}{4500}. \tag{3.14}$$

Substituting in (3.13), we get:

$$\alpha_1 = -\frac{4598}{125}. \tag{3.15}$$

Equations (3.14) and (3.15) imply that $(\alpha_1, \beta_1) \approx (-36.78, 0.51)$. Therefore, the ask price of the option is given by:

$$\alpha_0 = V_0(\alpha_1, \beta_1) \approx -36.78 + 0.51 \times 100 \times (1 - 10\%) = 9.12.$$

The results are summarized in the following diagram.

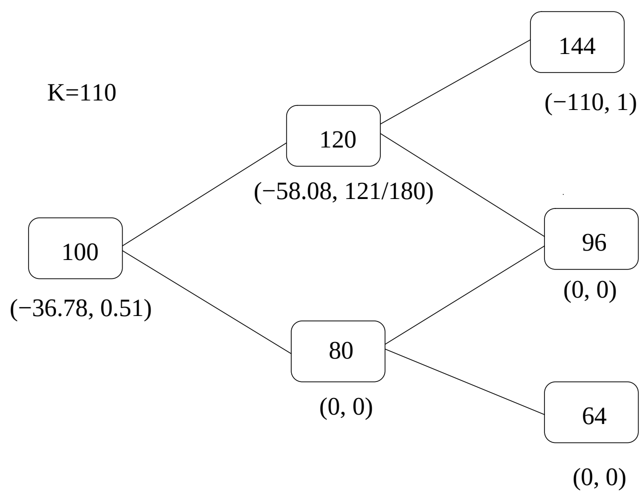


Figure 3.3: Hedging strategy obtained for a long call position

Proof of the Theorem. First, let us prove by backward induction that for each $t = 1, ..., T$, we have

$$\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u).$$

Consider the porfolio of the option at maturity. We have then two possibilities: either the call option is exercised, *i.e* $S_T > K$, in which case $(\alpha_{T+1}, \beta_{T+1}) = (-K, 1)$, or the option is not exercised, *i.e* $S_T < K$, in which case $(\alpha_{T+1}, \beta_{T+1}) = (0, 0)$. Since $S_T(u) > S_T(d)$, this implies that $\beta_{T+1}(u) \geq \beta_{T+1}(d)$. There are three possible cases for the portfolio (β_T, α_T):

- Case 1: If $K < S_T(d) < S_T(u)$, then $\beta_{T+1}(u) = \beta_{T+1}(d) = 1$. Therefore, we have $\beta_T = \beta_{T+1}(u)$ and $\alpha_T = -K$ which is the unique solution. It is straightforward that

$$\beta_{T+1}(d) \leq \beta_T \leq \beta_{T+1}(u).$$

- Case 2: If $S_T(d) < K < S_T(u)$, then we have

$$(\alpha_{T+1}(u), \beta_{T+1}(u)) = (-K, 1),$$
$$(\alpha_{T+1}(d), \beta_{T+1}(d)) = (0, 0).$$

  Therefore, equations (3.3) and (3.4) can be respectively rewritten as:

$$\begin{aligned}\beta_T S_{T-1}\hat{u} + \alpha_T =& S_{T-1}\hat{u} - K + k|\beta_T - 1|S_{T-1}\hat{u},\\ \beta_T S_{T-1}\hat{d} + \alpha_T =& k|\beta_T|S_{T-1}\hat{d}.\end{aligned}$$

  Following [BV92], we assume that $\beta_t \in (0, 1)$ for all t. This is a natural assumption in the hedging of long calls in models without friction, and simplifies our solution while ensuring that it exists. Hence, for any t, we have $0 \leq \beta_t \leq 1$ and the precedent equations imply:

$$\begin{aligned}\beta_T S_{T-1}\hat{u} + \alpha_T &= S_{T-1}\hat{u} - K + k(1 - \beta_T)S_{T-1}\hat{u},\\ \beta_T S_{T-1}\hat{d} + \alpha_T &= k\beta_T S_{T-1}\hat{d}.\end{aligned}$$

  By substracting the first one from the last one and after simplifications, we get:

$$\beta_T = \frac{S_{T-1}\hat{u}(1 + k) - K}{S_{T-1}\hat{u}(1 + k) - S_{T-1}\hat{d}(1 - k)},$$

  which represents the unique solution. One can easily see that we still have

$$\beta_{T+1}(d) \leq \beta_T \leq \beta_{T+1}(u).$$

- Case 3: If $S_T(d) < S_T(u) < K$, then $\beta_{T+1}(u) = \beta_{T+1}(d) = 0$. We thus have $\beta_T = 0$ and $\alpha_T = 0$, which is the unique solution. It is also true that we have the inequality

$$\beta_{T+1}(d) \leq \beta_T \leq \beta_{T+1}(u).$$

As a consequence, at the maturity time $t = T$, we have:

$$\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u). \tag{3.16}$$

Let us suppose that assumption (3.16) is true for $T-1, ..., t+1$ and prove that it also holds for t, where t is a non-terminal node of the binomial tree.

We then have

$$\begin{aligned} \beta_{t+2}(u,d) \leq \beta_{t+1}(u) \leq \beta_{t+2}(u,u), \\ \beta_{t+2}(d,d) \leq \beta_{t+1}(d) \leq \beta_{t+2}(u,d). \end{aligned}$$

Hence,

$$\beta_{t+1}(d) \leq \beta_{t+1}(u).$$

Let us now subtract (3.4) from (3.3) and transfer everything to the right-hand side. We get:

$$\begin{aligned} \beta_t S_{t-1}(\hat{u} - \hat{d}) + \beta_{t+1}(d) S_{t-1}\hat{d} - \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) + k|\beta_t - \beta_{t+1}(d)| S_{t-1}\hat{d} \\ - k|\beta_t - \beta_{t+1}(u)| S_{t-1}\hat{u} = 0. \end{aligned}$$

Let f be the function defined as:

$$\begin{aligned} f(\beta_t) :=& \beta_t S_{t-1}(\hat{u} - \hat{d}) + \beta_{t+1}(d) S_{t-1}\hat{d} - \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) \\ &+ k|\beta_t - \beta_{t+1}(d)| S_{t-1}\hat{d} - k|\beta_t - \beta_{t+1}(u)| S_{t-1}\hat{u} = 0. \end{aligned} \tag{3.17}$$

Three cases may occur:

- if $\beta_t < \beta_{t+1}(d)$, then

$$\begin{aligned} f(\beta_t) =& \beta_t S_{t-1}(\hat{u} - \hat{d}) + \beta_{t+1}(d) S_{t-1}\hat{d} - \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) \\ &+ k S_{t-1}(\beta_{t+1}(d)\hat{d} - \beta_{t+1}(u)\hat{u}) + k\beta_t S_{t-1}(\hat{u} - \hat{d}). \end{aligned}$$

- if $\beta_{t+1}(d) < \beta_t < \beta_{t+1}(u)$, then

$$\begin{aligned} f(\beta_t) =& \beta_t S_{t-1}(\hat{u} - \hat{d}) + \beta_{t+1}(d) S_{t-1}\hat{d} - \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) \\ &+ k\beta_t S_{t-1}(\hat{u} + \hat{d}) - k S_{t-1}(\beta_{t+1}(u)\hat{u} + \beta_{t+1}(d)\hat{d}). \end{aligned}$$

- if $\beta_{t+1}(u) < \beta_t$, then

$$\begin{aligned} f(\beta_t) =& \beta_t S_{t-1}(\hat{u} - \hat{d}) + \beta_{t+1}(d) S_{t-1}\hat{d} - \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) \\ &+ k\beta_t S_{t-1}(-\hat{u} + \hat{d}) - k S_{t-1}(\beta_{t+1}(u)\hat{u} + \beta_{t+1}(d)\hat{d}). \end{aligned}$$

Therefore, f has constant derivatives

$$f'(\beta_t) = \begin{cases} [(1+k)\hat{u} - (1+k)\hat{d}]S_{t-1} & \text{if } \beta_t \le \beta_{t+1}(d), \\ [(1+k)\hat{u} - (1-k)\hat{d}]S_{t-1} & \text{if } \beta_{t+1}(d) \le \beta_t \le \beta_{t+1}(u), \\ [(1-k)\hat{u} - (1-k)\hat{d}]S_{t-1} & \text{if } \beta_t \ge \beta_{t+1}(u). \end{cases}$$

In other words, f is piece-wise affine. Moreover, all its derivatives are positive numbers, that is, f is a piece-wise monotone increasing function. As a result, its root β_t is unique.

Now, let us prove that $\beta_{t+1}(d) \le \beta_t \le \beta_{t+1}(u)$. It is enough to show that we have

$$\begin{aligned} f(\beta_{t+1}(d)) &\le 0, \\ f(\beta_{t+1}(u)) &\ge 0. \end{aligned}$$

Using the induction assumption (3.16), we get:

$$\begin{aligned} f(\beta_{t+1}(d)) =& (\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u} - \alpha_{t+1}(u) + \alpha_{t+1}(d) + k(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u} \\ =& (1+k)(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u} - \alpha_{t+1}(u) + \alpha_{t+1}(d). \end{aligned}$$

Similarly:

$$\begin{aligned} f(\beta_{t+1}(u)) =& (\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) + k(\beta_{t+1}(u) - \beta_{t+1}(d))S_{t-1}\hat{d} \\ =& (1-k)(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d). \end{aligned}$$

Assumption (3.16) implies that

$$\begin{aligned} \beta_{t+2}(u,d) \le \beta_{t+1}(u) \le \beta_{t+2}(u,u), \\ \beta_{t+2}(d,d) \le \beta_{t+1}(d) \le \beta_{t+2}(u,d). \end{aligned}$$

According to the self-financing assumption:

$$\beta_{t+1}(u)S_{t-1}\hat{u}\hat{d} + \alpha_{t+1}(u) = \beta_{t+2}(u,d)S_{t-1}\hat{u}\hat{d} + \alpha_{t+2}(u,d) + k(\beta_{t+1}(u) - \beta_{t+2}(u,d))S_{t-1}\hat{u}\hat{d}, \tag{3.18}$$

$$\beta_{t+1}(d)S_{t-1}\hat{d}\hat{u} + \alpha_{t+1}(d) = \beta_{t+2}(u,d)S_{t-1}\hat{d}\hat{u} + \alpha_{t+2}(u,d) + k(\beta_{t+2}(u,d) - \beta_{t+1}(d))S_{t-1}\hat{d}\hat{u}. \tag{3.19}$$

Substracting (3.19) from (3.18), we obtain:

$$\begin{aligned} (\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{d}\hat{u} + \alpha_{t+1}(d) - \alpha_{t+1}(u) = \\ - kS_{t-1}\hat{d}\hat{u}[(\beta_{t+1}(u) - \beta_{t+2}(u,d)) - (\beta_{t+2}(u,d) - \beta_{t+1}(d))]. \end{aligned} \tag{3.20}$$

The lack of arbitrage implies that $\hat{d} < 1$. Consequently, we get:

$$\begin{aligned} f(\beta_{t+1}(d)) &= (1+k)(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &\le (1+k)(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d). \end{aligned}$$

Hence, equation (3.20) implies:

$$\begin{aligned}
f(\beta_{t+1}(d)) \leq & k(\beta_{t+1}(d)-\beta_{t+1}(u))S_{t-1}\hat{d}\hat{u}+[(\beta_{t+1}(d)-\beta_{t+1}(u))S_{t-1}\hat{d}\hat{u}-\alpha_{t+1}(u)+\alpha_{t+1}(d)] \\
\leq & k(\beta_{t+1}(d)-\beta_{t+1}(u))S_{t-1}\hat{u}\hat{d}-kS_{t-1}\hat{u}\hat{d}[(\beta_{t+1}(u)-\beta_{t+2}(u,d)) \\
& -(\beta_{t+2}(u,d)-\beta_{t+1}(d))] \\
\leq & -2kS_{t-1}\hat{u}\hat{d}[\beta_{t+1}(u)-\beta_{t+2}(u,d)] \leq 0.
\end{aligned}$$

Similarly, $f(\beta_{t+1}(u)) \geq 0$.

As a result, for $t = 1, ..., T$, we have:

$$\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u).$$

In particular,

$$\beta_2(d) \leq \beta_1 \leq \beta_2(u),$$

which completes the proof of the theorem. □

3.2 Replication of a Short Call Position

In the case of a short call position, Boyle and Vorst used a similar method, but assumed additional conditions, as stated in the following theorem.

Theorem 3.2.1. *By hedging the short call position at each time step, equations (3.3) and (3.4) have a unique solution* (α_t, β_t) *if:*

$$\hat{u}(1-k) \geq 1+k \tag{3.21}$$

and

$$\hat{d}(1+k) \leq 1-k. \tag{3.22}$$

Moreover, if

$$S_T(\omega) \notin (\frac{K}{1+k}, \frac{K}{1-k}), \tag{3.23}$$

for all $\omega \in \Omega$*, then*

$$\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d), \tag{3.24}$$

for $t = 1, \ldots, T$.

Remark 3.2.2. Observe that if equation (3.24) holds, then we have:

$$\begin{aligned}
(1-k)\beta_t S_{t-1}\hat{u}+\alpha_t =& (1-k)\beta_{t+1}(u)S_{t-1}\hat{u}+\alpha_{t+1}(u), \\
(1+k)\beta_t S_{t-1}\hat{d}+\alpha_t =& (1+k)\beta_{t+1}(d)S_{t-1}\hat{d}+\alpha_{t+1}(d),
\end{aligned}$$

that is,

$$\alpha_t-\alpha_{t+1}(u)+(\beta_t-\beta_{t+1}(u))^{+}S_{t-1}\hat{u}(1-k) = 0, \tag{3.25}$$

$$\alpha_t-\alpha_{t+1}(d)-(\beta_t-\beta_{t+1}(d))^{-}S_{t-1}\hat{d}(1+k) = 0. \tag{3.26}$$

Using the same argument as in Remark 3.1.2, one can see that equations (3.25) and (3.26) mean that, in the up-state, the additional amount of cash comes from selling $\beta_t - \beta_{t+1}(u)$ units of shares, whereas in the down-state, the debt in the cash flow is caused by the purchase of $\beta_{t+1}(d) - \beta_t$ units of shares.

Hence, Boyle Vorst's algorithm for the short call option works exactly as the same way as the case of the long position, except that equations (3.6) and (3.7) are replaced by equations (3.25) and (3.26).

Example 3.2.3. Consider the same binomial tree as in Example 3.1.3, with the same value $k = 10\%$.

Using Remark 3.2.2, we have:

$$(1-10\%) \times 144 \times \beta_2(u) + \alpha_2(u) = -(1-10\%) \times 144 \times 1 + 110, \tag{3.27}$$
$$(1+10\%) \times 96 \times \beta_2(u) + \alpha_2(u) = 0. \tag{3.28}$$

Subtracting equation (3.27) from (3.28) and solving for $\beta_2(u)$, we get:

$$\beta_2(u) = \frac{-(1-10\%) \times 144 + 110}{(1-10\%) \times 144 - (1+10\%) \times 96} = -\frac{49}{60}.$$

Substituting in equation (3.28), we obtain $\alpha_2(u) = 86.24$.

Similarly, we have

$$(1-10\%) \times 96 \times \beta_2(d) + \alpha_2(d) = 0, \tag{3.29}$$
$$(1+10\%) \times 64 \times \beta_2(d) + \alpha_2(d) = 0, \tag{3.30}$$

Substracting equation (3.29) from (3.30), we have $\beta_2(d) = 0$. By substitution in (3.29), we find that $\alpha_2(d) = 0$.

At time $t = 0$, we have:

$$(1-10\%) \times 120 \times \beta_1 + \alpha_1 = (1+10\%) \times 120 \times (-\frac{49}{60}) + 86.24, \tag{3.31}$$
$$(1+10\%) \times 80 \times \beta_1 + \alpha_1 = 0. \tag{3.32}$$

Subtracting equation (3.31) from (3.32) and solving for β_1, we get:

$$\beta_1 = \frac{(1-10\%) \times 120 \times (-\frac{49}{60}) + 86.24}{(1-10\%) \times 120 - (1+10\%) \times 80} = -\frac{49}{500}. \tag{3.33}$$

Substituting in (3.32), we have

$$\alpha_1 = \frac{1078}{125}. \tag{3.34}$$

Hence, equations (3.33) and (3.34) imply that $(\alpha_1, \beta_1) \approx (8.62, -0.10)$. It follows that the bid price of the option is given by

$$\alpha_0 = V_0(\alpha_1, \beta_1) \approx 8.62 - 0.10 \times 100 \times (1+10\%) = -2.38.$$

The results are summarized in Figure 3.4.

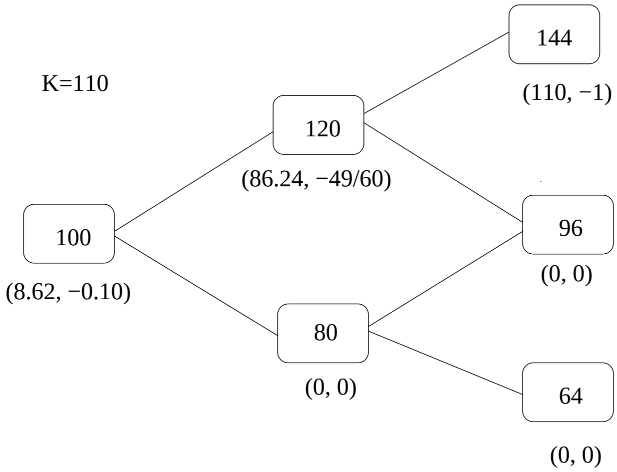


Figure 3.4: Hedging strategy obtained for a short call position

Proof. The proof of Theorem 3.2.1 is similar to Theorem 3.1.1 and and has been put in the Appendix. ☐

Remark 3.2.4. In both cases of a long and a short call position, Boyle and Vorst used the same algorithm and the no-abitrage argument [Rou06b] implying

$$\hat{d} < 1 < \hat{u}.$$

However, for the short call position, they had to add extra-conditions which are:

1. $\hat{u}(1-k) \geq 1+k$,
2. $\hat{d}(1+k) \leq 1-k$,
3. For all $\omega \in \Omega$, we have $S_T(\omega) \notin (\frac{K}{1+k}, \frac{K}{1-k})$.

The two first conditions ensure that the replicating portfolio is unique, whereas the third one implies that $\beta_2(u) \leq \beta_1 \leq \beta_2(d)$ if $\beta_2(u) \leq \beta_2(d)$.

In fact, the condition that β_t lies between $\beta_{t+1}(u)$ and $\beta_{t+1}(d)$ is only used for approximation results and is not compulsory. As a matter of fact, Palmer [Pal01] extended Boyle-Vorst's method and obtained an algorithm which works even if β_t does not lie between $\beta_{t+1}(u)$ and $\beta_{t+1}(d)$ and which gives results that are consistent with Boyle-Vorst's approach.

4. Roux-Tokarz-Zastawniak Algorithm

In this section, we will consider a method to super-replicate European options under transaction costs.

Roux, Tokarz and Zastawniak [RZ06] developed an algorithm to compute the price of European options using a concave function

$$Z_0 : x \mapsto \inf_{(\alpha,\beta)\in\mathcal{F}_{\text{writer}}} \alpha_1 + x\beta_1,$$

where $\mathcal{F}_{\text{writer}}$ is the set of all portfolios (α, β) which super-replicate the option writer's position. A portfolio (α_0, β_0) is sufficient to hedge the option writer's position if and only if $\alpha_0 + x\beta_0 \geq Z_0^x$ for all $x \in \mathbb{R}$. Hence, $\max_{x\in\mathbb{R}} Z_0^x$ represents the smallest amount of cash a writer with no stock holdings at time $t = 0$ would need to meet his liability. In other words, if $\beta_0 = 0$, the writer would have to invest at least $\max_{x\in\mathbb{R}} Z_0^x$ to hedge his position, that is $\alpha_0 \geq \max_{x\in\mathbb{R}} Z_0^x$. Therefore, $\max_{x\in\mathbb{R}} Z_0^x$ is an optimal price for the option's buyer and will constitute the ask price of the option.

4.1 Concave Functions

Definition 4.1.1. A function $f : \mathbb{R} \to \mathbb{R}\bigcup\{\infty\}$ is said to be *proper concave* if:

1. for all x,$z \in \mathbb{R}$, for all $\lambda \in (0, 1)$, $f(\lambda x + (1 - \lambda)z) \geq \lambda f(x) + (1 - \lambda)f(z)$;
2. There exists some $x \in \mathbb{R}$ such that $f(x) > -\infty$.

The *effective domain* of such a function is the set $dom\ f$ defined as:

$$dom\ f = \{x \in \mathbb{R}|f(x) > -\infty\}.$$

Definition 4.1.2. A function $f : \mathbb{R} \to \mathbb{R}\bigcup\{\infty\}$ is said to be *polyhedral proper concave* if:

1. $dom\ f$ is a closed set;
2. There exists $(a_i)_{0\leq i\leq n} \subseteq \mathbb{R}$ and $(b_j)_{0\leq j\leq n} \subseteq \mathbb{R}$ such that $f(x) = \min_{i=1,\dots,n}(a_i x + b_i)$.

Definition 4.1.3. The *concave cap* of functions $f_1, ..., f_n : \mathbb{R} \to \mathbb{R}\bigcup\{\infty\}$, denoted as $cap\{f_1, ..., f_n\}$, is the smallest concave function h such that $h \geq f_i$ for all $0 \leq i \leq n$.

We will admit the following result, referring to [TRZ06] or [Rou06a] for more details.

Lemma 4.1.4. *If $f_1, ..., f_n$ are proper polyhedral proper concave functions with bounded domains, then so is $cap\{f_1, ..., f_n\}$. Besides, we have:*

$$cap\{f_1, ..., f_n\} = \max_{((\lambda_1,x_1),\dots,(\lambda_n,x_n))\in A(x)} \sum_{i=1}^{n} \lambda_i f_i(x_i), \tag{4.1}$$

where

$$A(x) = \{((\lambda_1, x_1), ..., (\lambda_n, x_n))| \textit{ for any } i, \textit{ we have } \lambda_i \geq 0 \textit{ and } x_i \in dom\ f_i, \textit{and:} \sum_{i=1}^{n} \lambda_i = 1 \textit{ and } \sum_{i=1}^{n} \lambda_i x_i = x\}.$$

Notations

For any $x \in dom\ f$, let us denote by $D^+ f(x)$ (respectively $D^- f(x)$) the right (respectively the left) derivative of f at x.

We will adopt the following conventions:

$$\begin{array}{ll} D^- f(x) = +\infty & \text{if } f(y) = -\infty \text{ for all } y < x, \\ D^+ f(x) = -\infty & \text{if } f(y) = -\infty \text{ for all } y > x. \end{array}$$

The concavity of f implies that $D^+ f(x) \leq D^- f(x)$, for any $x \in dom\ f$.

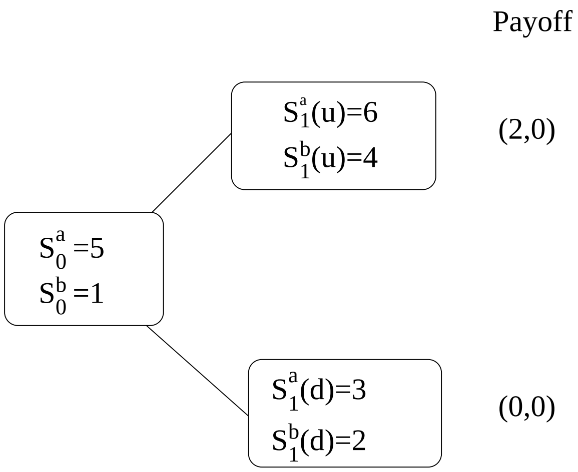
4.2 Ask Price Algorithm

We consider a European call option with payoff (ξ, ζ) and exercise time T. To compute the ask price of such an option, we construct adapted processes $Z = (Z_t)_{0 \leq t \leq T}$ and $(\tilde{Z})_{0 \leq t \leq T}$ as follows:

- We take
$$Z_T^x = \tilde{Z}_T^x = \begin{cases} \xi + x\zeta \text{ if } x \in [S_T^b, S_T^a], \\ -\infty \text{ if } x \notin [S_T^b, S_T^a]. \end{cases} \tag{4.2}$$
Hence, there is a function $Z_T(\mu) : x \mapsto Z_T^x(\mu)$ at each terminal node μ.
- For $t = 0, 1, ..., T-1$, we define $\tilde{Z}_t(\mu)$ as:
$$\tilde{Z}_t(\mu) = cap\{Z_{t+1}(v) \text{ such that } v \in \Omega_{t+1} \text{ is a successor node of } \mu\}, \tag{4.3}$$
where
$$Z_t^x = \begin{cases} \tilde{Z}_t^x \text{ if } x \in [S_t^b, S_t^a], \\ -\infty \text{ if } x \notin [S_t^b, S_t^a]. \end{cases} \tag{4.4}$$
- The ask price is computed as the maximum of Z_0, *i.e* $\max\limits_{x \in \mathbb{R}} Z_0^x$.

Remark 4.2.1. From Lemma 4.1.4, one can see that for any $t = 0, \ldots, T$ and $\omega \in \Omega$, the functions $Z_t(\omega)$ are polyhedral proper concave with bounded domains.

Remark 4.2.2. The lack of arbitrage ensures that $|\max\limits_{x \in \mathbb{R}} Z_0^x| < \infty$. As a matter of fact, the absence of arbitrage implies the existence of a risk-neutral probability P equivalent to the finite probability Q.



Example 4.2.3. To get an idea of how this algorithm works, let us consider Example 1.2.6: Following the RTZ algorithm, we have:

- At time $t = 1$, we have:

$$Z_1^x(u) = 2 + 0 \times x = 2 \text{ for } 4 \le x \le 6,$$
$$Z_1^x(d) = 0 + 0 \times x = 0 \text{ for } 2 \le x \le 3.$$

- At time $t = 0$, we simply take the concave cap of $\{Z_1(u), Z_1(d)\}$. We get:

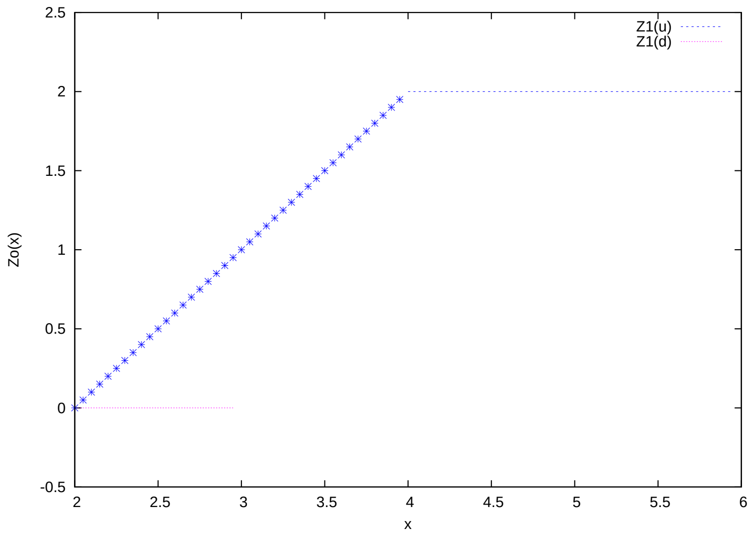


Figure 4.1: Graph of $Z_0(x)$

- The ask price is computed as $\max_{x \in \mathbb{R}} Z_0^x$, that is 2, which is consistent with the results obtained in Example 1.2.6.

4.3 Super-Hedging Strategy

The following lemma provides a way to build a self-financing strategy which hedges the long call position.

Lemma 4.3.1. *Let γ and δ be $\mathcal{F}_t$-measurable random variables such that $\gamma + x\delta \geq Z_t^x$ for any $x \in \mathbb{R}$ and any $t = 0, ..., T-1$.*
From γ and δ, it is possible to construct $\mathcal{F}_t$-measurable random variables ρ and σ such that, for any $x \in \mathbb{R}$:

$$\rho + x\sigma \geq Z_{t+1}^x, \tag{4.5}$$

$$V_t(\gamma - \rho, \delta - \sigma) \geq 0. \tag{4.6}$$

Proof. We have $\gamma + x\delta \geq Z_t^x$, for each $x \in \mathbb{R}$, that is, $\gamma \geq Z_t^x - x\delta$. Hence,

$$\gamma \geq \max_{x\in\mathbb{R}} Z_t^x - x\delta.$$

Since Z_t is a polyhedral proper concave with a bounded domain, the maximum is attained. Let thus y be an $\mathcal{F}_t$-measurable random variable such that

$$Z_t^y - y\delta = \max_{x\in\mathbb{R}} Z_t^x - x\delta. \tag{4.7}$$

Since the maximum is attained, $y \in [S_t^b, S_t^a]$ and therefore, $Z_t^y = \tilde{Z}_t^y > -\infty$.

To prove the lemma, it is enough to find σ and ρ such that (4.6) holds and

$$D^+\tilde{Z}_t^y \leq \sigma \leq D^-\tilde{Z}_t^y, \tag{4.8}$$

$$\rho + y\sigma = \tilde{Z}_t^y. \tag{4.9}$$

Because of the concavity of Z_t, equations (4.8) and (4.9) will imply that $\rho + x\sigma \geq Z_t^x$, for any $x \in \mathbb{R}$.

Let us prove the existence of such ρ and σ by analyzing all possible cases:

- Case 1: If $S_t^b < y < S_t^a$, then we have $D^-Z_t^y = D^-\tilde{Z}_t^y$, and $D^+Z_t^y = D^+\tilde{Z}_t^y$. Equation (4.7) implies that $D^+Z_t^y \leq \delta \leq D^-Z_t^y$. We take $\sigma = \delta$ and $\rho = Z_t^y - y\sigma$. It follows that (4.9) is true.

  Besides, we then have:

  $$\begin{aligned} \rho + y\sigma &= Z_t^y \leq \gamma + y\delta, \\ \sigma &= \delta. \end{aligned}$$

  Therefore, $(\gamma - \rho) + y(\delta - \sigma) \geq 0$, which implies (4.6).

- Case 2: If $y = S_t^b < S_t^a$, then we have $D^+Z_t^y = D^+\tilde{Z}_t^y$ and (4.7) implies that $D^+Z_t^y \leq \delta$. We take $\sigma = \min\{\delta, D^-\tilde{Z}_t^y\}$ and put $\rho = Z_t^y - y\sigma$. As a result, equation (4.9) holds.

  Besides, we have:

  $$\begin{aligned} \rho + y\sigma &= Z_t^y \leq \gamma + y\delta, \\ \sigma &\leq \delta, \end{aligned}$$

  that is, $(\gamma - \rho) + y(\delta - \sigma) \geq 0$. Since $y = S_t^b$, equation (4.6) is true.

- Case 3: If $y = S_t^a > S_t^b$, then we have $D^- Z_t^y = D^- \tilde{Z}_t^y$ and (4.7) implies that $\delta \leq D^- Z_t^y$. We take $\sigma = \max\{\delta, D^+ \tilde{Z}_t^y\}$ and $\rho = Z_t^y - y\sigma$, so that (4.9) holds.

  In this case, we have:
  $$\begin{aligned} \rho + y\sigma &= Z_t^y \leq \gamma + y\delta, \\ \sigma &\geq \delta, \end{aligned}$$
  that is, $(\gamma - \rho) + y(\delta - \sigma) \geq 0$. Since $y = S_t^a$, equation (4.6) holds.

- Case 4: If $y = S_t^b = S_t^a$, we take any finite value σ such between $D^+ \tilde{Z}_t^y$ and $D^- \tilde{Z}_t^y$ and put $\rho = Z_t^y - y\sigma$. Therefore, equation (4.9) holds.

  Moreover, we then have $\rho + y\sigma = Z_t^y \leq \gamma + y\delta$, that is, $(\gamma - \rho) + y(\delta - \sigma) \geq 0$. Since $y = S_t^a = S_t^b$, this implies that equation (4.6) holds.

□

4.3.1 Algorithm for the Hedging Strategy of the Writer's Position

Let (α, β) be the self-financing strategy that we will construct as the following:

- For $t = 0$, we put
  $$\begin{cases} \alpha_0 = \max_{x \in \mathbb{R}} Z_0^x, \\ \beta_0 = 0. \end{cases}$$
  It is clear that α_0 and β_0 are $\mathcal{F}_0$-measurable. Besides, for any $x \in \mathbb{R}$, $\alpha_0 + x\beta_0 \geq Z_0^x$.

- For $t = 1, ..., T-1$, if $\mathcal{F}_t$-measurable variables α_t, β_t are given, Lemma 4.3.1 gives a method to construct $\mathcal{F}_t$-measurable variables α_{t+1}, β_{t+1} such that $\alpha_{t+1} + x\beta_{t+1} \geq Z_{t+1}^x$, for any $x \in \mathbb{R}$.

- Since $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, α_{t+1} and β_{t+1} are also $\mathcal{F}_{t+1}$-measurable, and we iterate the procedure.

Therefore, (α, β) is a self-financing strategy such that for any $x \in [S_T^b, S_T^a]$:
$$\alpha_T + x\beta_T \geq Z_T^x = \xi + x\zeta,$$
that is,
$$(\alpha_T - \xi) + x(\beta_T - \zeta) \geq 0.$$
In other words, (α, β) super-replicates the option writer's position, and Definition 1.2.7 implies that
$$\pi^a(\xi, \zeta) \leq \alpha_0 = \max_{x \in \mathbb{R}} Z_0^x. \tag{4.10}$$
We will show in the following paragraph that we actually have $\pi^a(\xi, \zeta) = \alpha_0$.

4.3.2 Correctness of the Ask Price Algorithm

For $t = 1, ..., T$, $\mu \in \Omega_{t-1}$, $x \in dom\ Z_{t-1}(\mu)$, and $\nu \in succ\ \mu$, Lemma 4.1.4 ensures the existence of $\lambda_\nu \geq 0$ and $x_\nu \in dom\ Z_t(\nu)$ such that we have:

$$\sum_{\nu \in succ\ \mu} \lambda_\nu = 1, \tag{4.11}$$

$$\sum_{\nu \in succ\ \mu} \lambda_\nu x_\nu = x, \tag{4.12}$$

$$Z^x_{t-1}(\mu) = \sum_{\nu \in succ\ \mu} Z_t^{x_\nu} \lambda_\nu. \tag{4.13}$$

According to equation (4.11), if we define a measure $\hat{P}$ on Ω as:

$$\hat{P}(\nu|\mu) = \lambda_\nu,$$

then $\hat{P}$ is a probability measure, since $\lambda_\nu \geq 0$ for each $\nu \in succ\ \mu$.

Equations (4.11), (4.12) and (4.13) provide a way to construct adapted processes $\hat{S}$ and $\hat{Z}$ as follows:

- We put $\hat{Z}_0 = \max\limits_{x \in \mathbb{R}} Z_0^x = Z_0^{\hat{S}_0}$.
- For any $\nu \in succ\ \mu$, we set:

$$\hat{P}(\nu|\mu) = \lambda_\nu, \tag{4.14}$$

$$\hat{S}_{t-1}(\mu) = \sum_{\nu \in succ\ \mu} \lambda_\nu \hat{S}_t(\nu), \tag{4.15}$$

$$\hat{Z}_{t-1}(\mu) = \sum_{\nu \in succ\ \mu} \lambda_\nu \hat{Z}_t(\nu). \tag{4.16}$$

  Equations (4.15) and (4.16) imply that:

$$\hat{S}_{t-1}(\mu) = \sum_{\nu \in succ\ \mu} \hat{P}(\nu|\mu)\hat{S}_t(\nu) = \mathbb{E}_{\hat{P}}(\hat{S}_t|\mu),$$

$$\hat{Z}_{t-1}(\mu) = \sum_{\nu \in succ\ \mu} \hat{P}(\nu|\mu)\hat{Z}_t(\nu) = \mathbb{E}_{\hat{P}}(\hat{Z}_t|\mu).$$

  Therefore, $\hat{S}$ and $\hat{Z}$ are martingale processes under $\hat{P}$.

  In addition to the above properties, we take $\hat{S}$ and $\hat{Z}$ verifying:

$$S_t^b \leq \hat{S}_t \leq S_t^a, \tag{4.17}$$

$$\hat{Z}_t = Z_t^{\hat{S}_t}, \tag{4.18}$$

  for $t = 0, ..., T$.

Since $\hat{Z}$ is a martingale, we have:

$$\mathbb{E}_{\hat{P}}(\xi + \hat{S}_T\zeta) = \mathbb{E}_{\hat{P}}(\hat{Z}_T) = \hat{Z}_0 = \max_{x\in\mathbb{R}} Z_0^x. \tag{4.19}$$

The fact that $\hat{S}$ is a martingale implies that $(P, \hat{S}) \in \bar{\mathcal{P}}$, where $\bar{\mathcal{P}}$ is defined in Notation 2.2.3. As a result, according to equation (4.19), we have:

$$\max_{(P,S)\in\bar{\mathcal{P}}}(\xi + S_T\zeta) \geq \max_{x\in\mathbb{R}} Z_0^x, \tag{4.20}$$

Next, following Remark 1.2.9, let us take any strategy (α^a, β^a) super-replicating (ξ, ζ) such that $\pi^a(\xi, \zeta) = \alpha_0^a$. In addition, let us consider any $(P, S) \in \bar{\mathcal{P}}$.
Lemma 2.2.4 implies that $\alpha^a + \beta^a S$ is a supermartingale. Consequently:

$$\mathbb{E}_P(\alpha_T^a + \beta_T^a S_T) \leq \alpha_0^a. \tag{4.21}$$

Besides, since (α^a, β^a) super-replicates (ξ, ζ), we have:

$$V_T(\alpha_T^a - \xi, \beta_T^a - \zeta) = \alpha_T^a - \xi + [-(\beta_T^a - \zeta)^- S_T^a + (\beta_T^a - \zeta)^+ S_T^b] \geq 0. \tag{4.22}$$

Since $S_T^b \leq S_T \leq S_T^a$,

$$-(\beta_T^a - \zeta)^- S_T + (\beta_T^b - \zeta)^+ S_T \geq -(\beta_T^a - \zeta)^- S_T^a + (\beta_T^a - \zeta)^+ S_T^b. \tag{4.23}$$

Hence, equations (4.22) and (4.23) imply that

$$\alpha_T^a - \xi + [-(\beta_T^a - \zeta)^- S_T + (\beta_T^a - \zeta)^+ S_T] \geq 0,$$

that is,

$$\alpha_T^a + \beta_T^a S_T \geq \xi + \zeta S_T. \tag{4.24}$$

Using equation (4.24) and the fact $\alpha^a + +\beta^a S$ is a supermartingale, we obtain:

$$\mathbb{E}_P(\xi + \zeta S_T) \leq \mathbb{E}_P(\alpha_T^a + \beta_T^a S_T) \leq \alpha_0^a.$$

Since $(P, S) \in \bar{\mathcal{P}}$ is arbitrary, it follows that

$$\max_{(P,S)\in\bar{\mathcal{P}}} \mathbb{E}_P(\xi + \zeta S_T) \leq \mathbb{E}_P(\alpha_T^a + \beta_T^a S_T) \leq \alpha_0^a. \tag{4.25}$$

Equations (4.10), (4.20) and (4.25) give

$$\max_{x\in\mathbb{R}} Z_0^x \leq \max_{(P,S)\in\bar{\mathcal{P}}} \mathbb{E}_P(\xi + S_T\zeta) \leq \alpha_0^a = \pi^a(\xi, \zeta) \leq \max_{x\in\mathbb{R}} Z_0^x, \tag{4.26}$$

that is,

$$\pi^a(\xi, \zeta) = \max_{x\in\mathbb{R}} Z_0^x,$$

which completes the proof of the algorithm.

Conclusion

We discussed European option pricing in a binomial tree under proportional transaction costs on the risky asset. Working in the presence of transaction costs, buying or selling an option does not necessarily involves the same amount of cash. We thus defined the notions of ask and bid prices of an option, where the bid-ask spread represents the no-arbtitrage interval of the stock price. The ask price is obtained by hedging the option writer's positions, whereas the bid price is derived from the option buyer's positions.

Besides, we proved that under transaction costs, the absence of arbitrage amounts to the existence of a risk-neutral probability and an equivalent martingale measure.

We showed that in the presence of transaction costs, perfect hedging does not always work and we introduced the concept of super-replication. We then gave two examples of replicating and super-replicating algorithms: Boyle-Vorst's and Roux-Tokarz-Zastawniak's algorithms.

Boyle-Vorst's approach is based on the CRR binomial option. Their procedure consists in constructing the appropriate replicating portfolio at each trading date, including the transaction costs. However, Boyle and Vorst assumed the option to be exercised with actual delivery of the underlying asset, which is not the case in the Roux-Tokarz-Zastawniak's algorithm (RTZ algorithm).

The RTZ algorithm to price an option consists in computing the concave cap of some functions at each node, which leads to an optimal option price. The super-replicating strategy is then constructed, using an induction procedure starting at time $t = 0$. The RTZ algorithm works for arbitrary payoffs and in any discrete-time setting.

Appendix A. Proof of Theorems and Lemmas

Proof of Convexity Lemmas

Proof of Lemma 2.1.2. In order to prove that Φ is a convex set, it is enough to prove that:

1. for any $(\alpha,\beta),(\alpha',\beta')\in\Phi$, $(\alpha,\beta)+(\alpha',\beta')\in\Phi$;
2. for any $(\alpha,\beta)\in\Phi$ and $\lambda\geq 0$, $\lambda(\alpha,\beta)\in\Phi$.

Let $(\alpha,\beta)\in\Phi$. For $t=0,...,T$, we then have the following:

$$V_t(\alpha_t-\alpha_{t+1},\beta_t-\beta_{t+1})\geq 0,$$
$$V_t({\alpha'}_t-{\alpha'}_{t+1},{\beta'}_t-{\beta_{t+1}}')\geq 0.$$

That is,

$$(\alpha_t+{\alpha'}_t)-(\alpha_{t+1}+{\alpha_{t+1}}')+[(\beta_t-\beta_{t+1})^{+}+({\beta'}_t-{\beta_{t+1}}')^{+}]S_t^b \\ -[(\beta_t-\beta_{t+1})^{-}+({\beta'}_t-{\beta_{t+1}}')^{-}]S_t^a\geq 0, \quad \text{(A.1)}$$

By transferring some terms to the right-hand side of the inequality, we find another expression which allow us to prove that (A.1) implies $(\alpha,\beta)+(\alpha',\beta')\in\Phi$.

As a matter of fact, (A.1) is equivalent to say that

$$(\alpha_{t+1}+{\alpha_{t+1}}')-(\alpha_t+{\alpha'}_t) \\ \leq[(\beta_t-\beta_{t+1})^{+}+({\beta'}_t-{\beta_{t+1}}')^{+}]S_t^b-[(\beta_t-\beta_{t+1})^{-}+({\beta'}_t-{\beta_{t+1}}')^{-}]S_t^a. \quad \text{(A.2)}$$

Now, saying that $(\alpha,\beta)+(\alpha',\beta')=(\alpha+\alpha',\beta+\beta')$ is in Φ is equivalent to say that for any $t=0,...,T$, we have

$$(\alpha_t+{\alpha'}_t)-(\alpha_{t+1}+{\alpha_{t+1}}')+[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{+}S_t^b \\ -[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{-}S_t^a\geq 0,$$

To compare this equation to (A.2), let us move some terms to the right-hand side of the inequality. We get:

$$(\alpha_{t+1}+{\alpha_{t+1}}')-(\alpha_t+{\alpha'}_t) \\ \leq[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{+}S_t^b-[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{-}S_t^a. \quad \text{(A.3)}$$

Henceforth, to prove that (A.3) effectively holds, (A.2) implies that it is enough to prove that

$$[(\beta_t-\beta_{t+1})^{+}+({\beta'}_t-{\beta_{t+1}}')^{+}]S_t^b-[(\beta_t-\beta_{t+1})^{-}+({\beta'}_t-{\beta_{t+1}}')^{-}]S_t^a \\ \leq[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{+}S_t^b-[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^{-}S_t^a. \quad \text{(A.4)}$$

Let us denote:

$$\begin{aligned}A =&[(\beta_t-\beta_{t+1})^+ + ({\beta'}_t-{\beta_{t+1}}')^+]S_t^b - [(\beta_t-\beta_{t+1})^- + ({\beta'}_t-{\beta_{t+1}}')^-]S_t^a,\\ B =&[(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^+S_t^b - [(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]^-S_a^t.\end{aligned}$$

To prove that we indeed have $A \le B$, let us consider the possible cases.

- If $\beta_t-\beta_{t+1}\ge 0$ and ${\beta'}_t-{\beta_{t+1}}'\ge 0$, then:

$$\begin{aligned}A &= [(\beta_t-\beta_{t+1})+({\beta'}_t-{\beta_{t+1}}')]S_t^b\\ &= [(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')]S_t^b\\ &= [(\beta_t+{\beta'}_t)^+-(\beta_{t+1}+{\beta_{t+1}}')^+]S_t^b\\ &= B.\end{aligned}$$

- Similarly, if $\beta_t-\beta_{t+1}<0$ and ${\beta'}_t-{\beta_{t+1}}'<0$, we also have the equality $A=B$.
- If $\beta_t-\beta_{t+1}\ge 0$, ${\beta'}_t-{\beta_{t+1}}'<0$ and $(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')\ge 0$, then:

$$\begin{aligned}A &= (\beta_t-\beta_{t+1})S_t^b+({\beta'}_t-{\beta_{t+1}}')S_t^a,\\ B &= (\beta_t-\beta_{t+1})S_t^b+({\beta'}_t-{\beta_{t+1}}')S_t^b.\end{aligned}$$

  Since $S_t^a\ge S_t^b$, it follows that $A\le B$.
- Similarly, if $\beta_t-\beta_{t+1}<0$, ${\beta'}_t-{\beta_{t+1}}'\ge 0$ and $(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')\ge 0$, then $A\le B$.
- If $\beta_t-\beta_{t+1}\ge 0$, ${\beta'}_t-{\beta_{t+1}}'<0$ and $(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')<0$, then:

$$\begin{aligned}A &= (\beta_t-\beta_{t+1})S_t^b+({\beta'}_t-{\beta_{t+1}}')S_t^a,\\ B &= (\beta_t-\beta_{t+1})S_t^a+({\beta'}_t-{\beta_{t+1}}')S_t^a.\end{aligned}$$

  Since $S_t^b\ge S_t^a$, we have $A\le B$.
- Similarly, if $\beta_t-\beta_{t+1}<0$, ${\beta'}_t-{\beta_{t+1}}'\ge 0$ and $(\beta_t+{\beta'}_t)-(\beta_{t+1}+{\beta_{t+1}}')<0$, then we have $A\le B$.

Hence, equation (A.4) always holds and $(\alpha,\beta)+(\alpha',\beta')\in\Phi$.

Besides, it is clear that for any $(\alpha,\beta)\in\Phi$ and any $\lambda\ge 0$, $\lambda(\alpha,\beta)=(\lambda\alpha,\lambda\beta)\in\Phi$. Indeed, since λ is positive, we have for $t=0,...,T-1$:

$$\begin{aligned}V_t(\lambda[\alpha_t-\alpha_{t+1}],\lambda[\beta_t-\beta_{t+1}]) =&\lambda(\alpha_t-\alpha_{t+1})-[\lambda(\beta_t-\beta_{t+1})]^-S_t^a+[\lambda(\beta_t-\beta_{t+1})]^+S_t^b\\ =&\lambda(\alpha_t-\alpha_{t+1})-\lambda[\beta_t-\beta_{t+1}]^-S_a^t+\lambda[\beta_t-\beta_{t+1}]^+S_t^b\\ =&\lambda[(\alpha_t-\alpha_{t+1})-(\beta_t-\beta_{t+1})^-S_t^a+(\beta_t-\beta_{t+1})^+S_t^b]\\ =&\lambda V_t(\alpha_t-\alpha_{t+1},\beta_t-\beta_{t+1})\ge 0.\end{aligned}$$

Besides, $\lambda\alpha_0 = 0$.

Therefore, $\lambda(\alpha, \beta) \in \Phi$ and it follows that Φ is a convex cone.

Finally, Φ is defined by a system of weak inequalities between continuous functions, which implies that Φ is a closed set. □

Proof of Lemma 2.1.4. To prove that $\mathcal{M}$ is a closed convex cone, it is sufficient to prove that:

1. if $X, X' \in \mathcal{M}$, then $X + X' \in \mathcal{M}$;
2. if $X \in \mathcal{M}$ and $\lambda \geq 0$, then $\lambda X \in \mathcal{M}$.

Let $X, X' \in \mathcal{M}$. The definition of $\mathcal{M}$ implies that there exist $(\alpha, \beta), (\alpha', \beta') \in \Phi$ such that:

$$\alpha_0 + \alpha'_0 \leq 0, \tag{A.5}$$
$$(\alpha_{T+1} + \alpha'_{T+1}) - (\beta^-_{T+1} + ({\beta_{T+1}}')^-)S^a_T + (\beta^+_{T+1} + ({\beta_{T+1}}')^+)S^b_T \geq X + X'. \tag{A.6}$$

Hence, (A.6) implies that in order to prove that $X + X' \in \mathcal{M}$, it is enough to prove that

$$\begin{aligned}(\alpha_{T+1} + \alpha'_{T+1}) &- (\beta_{T+1} + {\beta_{T+1}}')^- S^a_T + (\beta_{T+1} + {\beta_{T+1}}')^+ S^b_T \\ &\geq (\alpha_{T+1} + \alpha'_{T+1}) - (\beta^-_{T+1} + ({\beta_{T+1}}')^-)S^a_T + (\beta^+_{T+1} + ({\beta_{T+1}}')^+)S^b_T,\end{aligned}$$

that is,

$$\begin{aligned}-(\beta^-_{T+1} &+ ({\beta_{T+1}}')^-)S^a_T + (\beta^+_{T+1} + ({\beta_{T+1}}')^+)S^b_T \\ &\leq -(\beta_{T+1} + {\beta_{T+1}}')^- S^a_T + (\beta_{T+1} + {\beta_{T+1}}')^+ S^b_T.\end{aligned} \tag{A.7}$$

Using a similar argument to the one used in Lemma 2.1.2, we find that (A.7) holds. Indeed, let:

$$\begin{aligned}A &= -(\beta^-_{T+1} + ({\beta_{T+1}}')^-)S^a_T + (\beta^+_{T+1} + ({\beta_{T+1}}')^+)S^b_T, \\ B &= -(\beta_{T+1} + {\beta_{T+1}}')^- S^a_T + (\beta_{T+1} + {\beta_{T+1}}')^+ S^b_T.\end{aligned}$$

Let us consider all the possible cases.

- If $\beta_{T+1} \geq$ and ${\beta_{T+1}}' \geq 0$, then $A = (\beta_{T+1} + {\beta_{T+1}}')S^b_T = (\beta_{T+1} + {\beta_{T+1}}')^+ S^b_T = B$.
- Similarly, if $\beta_{T+1} < 0$ and ${\beta_{T+1}}' < 0$, we also have the equality $A = B$.
- If $\beta_{T+1} \geq 0$, ${\beta_{T+1}}' < 0$ and $\beta_{T+1} + {\beta_{T+1}}' \geq 0$, then

  $$\begin{aligned}A &= \beta_{T+1}S^b_T + {\beta_{T+1}}'S^a_T, \\ B &= \beta_{T+1}S^b_T + {\beta_{T+1}}'S^b_T.\end{aligned}$$

  Since $S^b_T \leq S^a_T$, it follows that $A \leq B$.
- Likewise, if $\beta_{T+1} < 0$, ${\beta_{T+1}}' \geq 0$ and $\beta_{T+1} + {\beta_{T+1}}' \geq 0$, then $A \leq B$.

- If $\beta_{T+1} \geq 0$, ${\beta_{T+1}}' < 0$ and $\beta_{T+1} + {\beta_{T+1}}' < 0$, then

$$A = {\beta_{T+1}}' S_T^a + \beta_{T+1} S_T^b,$$
$$B = {\beta_{T+1}}' S_T^a + \beta_{T+1} S_T^a.$$

  Since $S_T^a \leq S_T^b$, it follows that $A \leq B$.

- Similarly, if $\beta_{T+1} < 0$, ${\beta_{T+1}}' \geq 0$ and $\beta_{T+1} + {\beta_{T+1}}' < 0$, we also have $A \leq B$.

Therefore, $(\alpha, \beta) + (\alpha', \beta') = (\alpha + \alpha', \beta + \beta')$ is self-financing. Besides, $\alpha_0 + {\alpha'}_0 = 0$ and $V_T(\alpha + \alpha', \beta + \beta') \geq X + X'$. Consequently, $X + X' \in \mathcal{M}$.

Let $X \in \mathcal{M}$ and $\lambda \geq 0$. Since $X \in \mathcal{M}$, there exists $(\alpha, \beta) \in \Phi$ such that

$$\alpha_0 \leq 0,$$
$$\alpha_{T+1} - \beta_{T+1}^- S_T^a + \beta_{T+1}^+ S_T^b \geq X.$$

Consider the strategy $\lambda(\alpha, \beta) = (\lambda\alpha, \lambda\beta)$. We have indeed

$$\lambda\alpha_0 \leq 0. \tag{A.8}$$

Besides, since $\lambda \geq 0$, one can see that:

$$\lambda\alpha_{T+1} - (\lambda\beta_{T+1})^- S_T^a + (\lambda\beta_{T+1})^+ S_T^b = \lambda[\alpha_{T+1} - \beta_{T+1}^- S_T^a + \beta_{T+1}^+ S_T^b] = \lambda V_T(\alpha_{T+1}, \beta_{T+1}) \geq \lambda X. \tag{A.9}$$

According to Lemma 2.1.2, $(\lambda\alpha, \lambda\beta) \in \Phi$. As a result, (A.8) and (A.9) imply that $\lambda X \in \mathcal{M}$. In conclusion, $\mathcal{M}$ is a convex set.

Using the same argument as in Lemma 2.1.2, one can conclude that $\mathcal{M}$ is closed. ☐

Proof of Boyle-Vorst's Theorem for the Short Position

Proof of Theorem 3.2.1. First, let us prove that β_t is unique for all t. Consider the same function f used in the proof corresponding to Theorem 3.1.1, that is:

$$\begin{aligned} f(\beta_t) :=& \beta_t S_{t-1}(\hat{u} - \hat{d}) - \beta_{t+1}(u) S_{t-1}\hat{u} + \beta_{t+1}(d) S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ & - k|\beta_t - \beta_{t+1}(u)| S_{t-1}\hat{u} + k|\beta_t - \beta_{t+1}(d)| S_{t-1}\hat{d} = 0. \end{aligned}$$

The function f is piece-wise linear. Indeed, if $\beta_{t+1}(u) \leq \beta_{t+1}(d)$, then we have the following cases:

- if $\beta_t \leq \beta_{t+1}(u)$, then

$$\begin{aligned} f(\beta_t) =& \beta_t S_{t-1}(\hat{u} - \hat{d}) - \beta_{t+1}(u) S_{t-1}\hat{u} + \beta_{t+1}(d) S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ & + k(\beta_t - \beta_{t+1}(u)) S_{t-1}\hat{u} + k(\beta_{t+1}(d) - \beta_t) S_{t-1}\hat{d}. \end{aligned}$$

- if $\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d)$, then

$$\begin{aligned} f(\beta_t) =&\beta_t S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &- k(\beta_t - \beta_{t+1}(u))S_{t-1}\hat{u} - k(\beta_t - \beta_{t+1}(d))S_{t-1}\hat{d}. \end{aligned}$$

- if $\beta_t \geq \beta_{t+1}(d)$, then

$$\begin{aligned} f(\beta_t) =&\beta_t S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &- k(\beta_t - \beta_{t+1}(u))S_{t-1}\hat{u} + k(\beta_t - \beta_{t+1}(d))S_{t-1}\hat{d}. \end{aligned}$$

Therefore, f' is given by:

$$f'(\beta_t) = \begin{cases} [\hat{u}(1+k) - \hat{d}(1+k)]S_{t-1} \text{ if } \beta_t \leq \beta_{t+1}(u), \\ [\hat{u}(1-k) - \hat{d}(1+k)]S_{t-1} \text{ if } \beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d), \\ [\hat{u}(1-k) - \hat{d}(1-k)]S_{t-1} \text{ if } \beta_t \geq \beta_{t+1}(d). \end{cases}$$

Similarly, let us consider the case where $\beta_{t+1}(u) > \beta_{t+1}(d)$. The function f is thus given by:

- if $\beta_t \leq \beta_{t+1}(d)$, then

$$\begin{aligned} f(\beta_t) =&\beta_t S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}u + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &+ k(\beta_t - \beta_{t+1}(u))S_{t-1}\hat{u} + k(\beta_{t+1}(d) - \beta_t)S_{t-1}\hat{d}. \end{aligned}$$

- if $\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u)$, then

$$\begin{aligned} f(\beta_t) =&\beta_t S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &+ k(\beta_t - \beta_{t+1}(u))S_{t-1}\hat{u} + k(\beta_t - \beta_{t+1}(d))S_{t-1}\hat{d}. \end{aligned}$$

- if $\beta_t \geq \beta_{t+1}(u)$, then

$$\begin{aligned} f(\beta_t) =&\beta_t S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\ &- k(\beta_t - \beta_{t+1}(u))S_{t-1}\hat{u} + k(\beta_t - \beta_{t+1}(d))S_{t-1}\hat{d}. \end{aligned}$$

Therefore, the function f has constant derivatives:

$$f'(\beta_t) = \begin{cases} [\hat{u}(1+k) - \hat{d}(1+k)]S_{t-1} \text{ if } \beta_t \leq \beta_{t+1}(d), \\ [\hat{u}(1+k) - \hat{d}(1-k)]S_{t-1} \text{ if } \beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u), \\ [\hat{u}(1-k) - \hat{d}(1-k)]S_{t-1} \text{ if } \beta_t \geq \beta_{t+1}(u). \end{cases}$$

Since $(1-k)\hat{u} \geq (1+k)\hat{d}$, we have $f' \geq 0$ and f is a monotone increasing and continuous function. As a consequence, it has a unique zero β_t.

Let us prove by induction that for all $t \leq T-1$, we have

$$\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d).$$

To start the induction, consider the portfolio at maturity. We then have three possible cases:

- Case 1: If $K < S_T(d)$, then $\beta_{T+1}(d) = \beta_{T+1}(u) = -1$ and $\alpha_{T+1}(d) = \alpha_{T+1}(u) = K$. Hence, $\beta_T = -1$. By substitution in (3.3) or (3.4), we get the corresponding unique solution for α_T. It is straightforward that the inequality $\beta_{T+1}(u) \leq \beta_T \leq \beta_{T+1}(d)$ holds in this case.

- Case 2: If $K > S_T(u)$, then $\beta_{T+1}(d) = \beta_{T+1}(u) = 0$ and $\alpha_{T+1}(u) = \alpha_{T+1}(d) = 0$, in which case $\beta_T = 0$. Therefore, $\alpha_T = 0$. We then have indeed $\beta_{T+1}(u) \leq \beta_T \leq \beta_{T+1}(d)$.

- Case 3: If $S_T(d) \leq K \leq S_T(u)$, then $\beta_{T+1}(u) = -1$ and $\beta_{T+1}(d) = 0$, and equations (3.3) and (3.4) give:

$$\begin{aligned}
\beta_T S_{T-1}\hat{u} + \alpha_T &= -S_{T-1}\hat{u} + K + k|\beta_T + 1|S_{T-1}\hat{u}, \\
\beta_T S_{T-1}\hat{d} + \alpha_T &= k|\beta_T|S_{T-1}\hat{d}.
\end{aligned}$$

Using a similar argument as in the case of the long call option, we assume that $\beta_t \in (0, -1)$ for each $t = 0, \ldots, T$. Hence, we have $|\beta_t| = -\beta_t$ and $|\beta_t + 1| = \beta_t + 1$. We get:

$$\begin{aligned}
(1-k)\beta_T S_{T-1}\hat{u} + \alpha_{T-1} &= -S_{T-1}\hat{u} + K + kS_{T-1}\hat{u}, \\
(1+k)\beta_T S_{T-1}\hat{d} + \alpha_{T-1} &= 0.
\end{aligned}$$

Substracting the first equation from the last one, we obtain:

$$\beta_T = -\frac{S_{T-1}\hat{u}(1-k) - K}{S_{T-1}[\hat{u}(1-k) - \hat{d}(1+k)]}. \tag{A.10}$$

If the assumption (3.23) holds, then $S_{T-1}\hat{d}(1+k) < K < S_{T-1}\hat{u}(1-k)$, and (A.10) implies:

$$-1 \leq \beta_{T-1} \leq 0.$$

Therefore, we have

$$\beta_{T+1}(u) \leq \beta_T \leq \beta_{T+1}(d).$$

Let us assume that this is true up to time $t+1$ and prove that this holds up to time t. By assumption, we have

$$\begin{aligned}
\beta_{t+2}(u,u) \leq \beta_{t+1}(u) \leq \beta_{T+2}(u,d), \\
\beta_{t+2}(u,d) \leq \beta_{t+1}(d) \leq \beta_{t+2}(d,d).
\end{aligned}$$

Henceforth,

$$\beta_{t+1}(u) \leq \beta_{t+1}(d). \tag{A.11}$$

Since f is monotone increasing, in order to prove that $\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d)$, it is enough to show that $f(\beta_{t+1}(u)) \leq 0$ and $f(\beta_{t+1}(d)) \geq 0$.

Equation (A.11) implies:

$$\begin{aligned}
f(\beta_{t+1}(d)) =& \beta_{t+1}(d)S_{t-1}(\hat{u} - \hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d) \\
& - k(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u} \\
=& S_{t-1}\hat{u}(\beta_{t+1}(d) - \beta_{t+1}(u))(1-k) - \alpha_{t+1}(u) + \alpha_{t+1}(d).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}f(\beta_{t+1}(u)) =&\beta_{t+1}(u)S_{t-1}(\hat{u}-\hat{d}) - \beta_{t+1}(u)S_{t-1}\hat{u} + \beta_{t+1}(d)S_{t-1}\hat{d} - \alpha_{t+1}(u) + \alpha_{t+1}(d)\\ &- k(\beta_{t+1}(u) - \beta_{t+1}(d))S_{t-1}\hat{d}\\ =&S_{t-1}\hat{d}(\beta_{t+1}(d) - \beta_{t+1}(u))(1+k) - \alpha_{t+1}(u) + \alpha_{t+1}(d).\end{aligned}$$

Using the self-financing assumption, we get:

$$\beta_{t+1}(u)S_{t-1}\hat{u}\hat{d} + \alpha_{t+1}(u) = \alpha_{t+2}(u,d) + \beta_{t+2}(u,d)S_{t-1}\hat{u}\hat{d} - k(\beta_{t+1}(u) - \beta_{t+2}(u,d))S_{t-1}\hat{u}\hat{d}, \tag{A.12}$$

$$\beta_{t+1}(d)S_{t-1}\hat{u}\hat{d} + \alpha_{t+1}(d) = \alpha_{t+2}(u,d) + \beta_{t+2}(u,d)S_{t-1}\hat{u}\hat{d} - k(\beta_{t+2}(u,d) - \beta_{t+1}(d))S_{t-1}\hat{u}\hat{d}. \tag{A.13}$$

Substracting equation (A.13) from (A.12), we obtain:

$$\begin{aligned}&(\beta_{t+1}(d) - \beta_{t+1}(u))S_{t-1}\hat{u}\hat{d} + \alpha_{t+1}(d) - \alpha_{t+1}(u)\\ &\quad = -kS_{t-1}\hat{u}\hat{d}[(\beta_{t+2}(u,d) - \beta_{t+1}(d)) - (\beta_{t+1}(u) - \beta_{t+2}(u,d))].\end{aligned} \tag{A.14}$$

The first assumption of the theorem implies that $\dfrac{\hat{u}(1-k)}{(1+k)} \geq 1$. Thus:

$$\begin{aligned}f(\beta_{t+1}(u)) &= S_{t-1}\hat{d}(\beta_{t+1}(d) - \beta_{t+1}(u))(1+k) - \alpha_{t+1}(u) + \alpha_{t+1}(d)\\ &\leq S_{t-1}\hat{u}\hat{d}(\beta_{t+1}(d) - \beta_{t+1}(u))(1-k) - \alpha_{t+1}(u) + \alpha_{t+1}(d).\end{aligned}$$

Using (A.14), we find that:

$$\begin{aligned}f(\beta_{t+1}(u)) \leq& -kS_{t-1}\hat{u}\hat{d}(\beta_{t+1}(d) - \beta_{t+1}(u)) - kS_{t-1}\hat{u}\hat{d}[(\beta_{t+2}(u,d) - \beta_{t+1}(d))\\ &- (\beta_{t+1}(u) - \beta_{t+2}(u,d))]\\ \leq& -2kS_{t-1}\hat{u}\hat{d}[\beta_{t+2}(u,d) - \beta_{t+1}(u)] \leq 0.\end{aligned}$$

Similarly, we prove that $f(\beta_{t+1}(d)) \geq 0$.

In conclusion, $\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d)$, which completes the proof of the theorem.

□

Appendix B. Palmer's Extension to the Boyle-Vorst's Algorithm

Palmer extended Boyle-Vorst's replicating method and obtained an algorithm which works for both long and short call positions and without any additional conditions for the short position. Using the same binomial approach, Palmer [Pal01] gives conditions under which the initial replicating portfolio is unique.
Recall that the self-financing argument implies:

$$\begin{aligned}\beta_t S_{t-1}\hat{u} + \alpha_t &= \beta_{t+1}(u) S_{t-1}\hat{u} + \alpha_{t+1}(u) + k|\beta_t - \beta_{t+1}(u)| S_{t-1}\hat{u},\\ \beta_t S_{t-1}\hat{d} + \alpha_t &= \beta_{t+1}(d) S_{t-1}\hat{d} + \alpha_{t+1}(d) + k|\beta_t - \beta_{t+1}(d)| S_{t-1}\hat{d}.\end{aligned}$$

Theorem B.0.2. *If one of the two followings assertions is true:*

1. $\beta_{t+1}(u) < \beta_{t+1}(d)$ *and* $k(\hat{u}+\hat{d})(\hat{u}-\hat{d}) < 1$;
2. $\beta_{t+1}(u) \geq \beta_{t+1}(d)$ *and* $k < 1$;

then equations (3.3) and (3.4) have a unique solution (α_t, β_t), *for* $t = 1, \ldots, T$. *In particular, there is then a unique initial portfolio* (α_1, β_1).

Proof. Substracting equation (3.3) by (3.4), we get:

$$\begin{aligned}S_{t-1}(\hat{u}-\hat{d})\beta_t =& \beta_{t+1}(u) S_{t-1}\hat{u} - \beta_{t+1}(d) S_{t-1}\hat{d} + \alpha_{t+1}(u) - \alpha_{t+1}(d) + k|\beta_{t+1}(u) - \beta_t| S_{t-1}\hat{u}\\ &- k|\beta_{t+1}(d) - \beta_t| S_{t-1}\hat{d}.\end{aligned}$$

Dividing by $S_{t-1}(\hat{u}-\hat{d})$ and putting everything to the left-hand side, we obtain the following function g defined as:

$$\begin{aligned}g(\beta_t) :=& \beta_t - \frac{\beta_{t+1}(u)\hat{u} - \beta_{t+1}(d)\hat{d}}{\hat{u}-\hat{d}} - \frac{\alpha_{t+1}(u) - \alpha_{t+1}(d)}{S_{t-1}(\hat{u}-\hat{d})}\\ &- k\frac{|\beta_{t+1}(u) - \beta_t|\hat{u} - |\beta_{t+1}(d) - \beta_t|\hat{d}}{\hat{u}-\hat{d}} = 0. \end{aligned} \tag{B.1}$$

The function g is thus a piece-wise affine.

Consider the function h defined as

$$h(\beta_t) := -k\frac{|\beta_{t+1}(u) - \beta_t|\hat{u} - |\beta_{t+1}(d) - \beta_t|\hat{d}}{\hat{u}-\hat{d}}.$$

Observe that we have:

$$g'(\beta_t) = 1 + h'(\beta_t). \tag{B.2}$$

If $\beta_{t+1}(u) < \beta_{t+1}(d)$, then the function h is given by:

- if $\beta_t \leq \beta_{t+1}(u)$, then

$$h(\beta_t) = -k\frac{(\beta_{t+1}(u) - \beta_t)\hat{u} - (\beta_{t+1}(d) - \beta_t)\hat{d}}{\hat{u} - \hat{d}};$$

- if $\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d)$, then

$$h(\beta_t) = -k\frac{(\beta_t - \beta_{t+1}(u))\hat{u} - (\beta_{t+1}(d) - \beta_t)\hat{d}}{\hat{u} - \hat{d}};$$

- if $\beta_t \geq \beta_{t+1}(d)$, then

$$h(\beta_t) = k\frac{(\beta_{t+1}(u) - \beta_t)\hat{u} - (\beta_{t+1}(d) - \beta_t)\hat{d}}{\hat{u} - \hat{d}}.$$

Hence, equation (B.2) implies that the corresponding values of g' are given by:

$$g'(\beta_t) = \begin{cases} 1 - k(-\hat{u} + \hat{d})/(\hat{u} - \hat{d}) = 1 + k, \\ 1 - k(\hat{u} + \hat{d})/(\hat{u} - \hat{d}), \\ 1 - k(\hat{u} - \hat{d})(\hat{u} - \hat{d}) = 1 - k. \end{cases}$$

Since

$$-k(\hat{u} + \hat{d})/(\hat{u} - \hat{d}) \leq -k \leq k,$$

we have

$$g'(\beta_t) \geq 1 - k(\hat{u} + \hat{d})/(\hat{u} - \hat{d}).$$

Assertion 1 implies that

$$k(\hat{u} + \hat{d})/(\hat{u} - \hat{d}) < 1.$$

Therefore, $g' > 0$ and g is monotone increasing.

Besides, $\lim_{x \to \pm\infty} g(x) = \pm\infty$. Since g is continuous, g has a zero, and this zero is unique. Solving for α_t by substituting in (3.3) or (3.4), we obtain the unique solution (α_t, β_t).

Similarly, let us consider the case where $\beta_{t+1}(d) \leq \beta_{t+1}(u)$. In this case, g is also a piece-wise continuous affine function. As a matter of fact, we get:

- if $\beta_t \leq \beta_{t+1}(d)$, then

$$h(\beta_t) = -k\frac{(\beta_{t+1}(u) - \beta_t)\hat{u} - (\beta_{t+1}(d) - \beta_t)\hat{d}}{\hat{u} - \hat{d}};$$

- if $\beta_{t+1}(d) \leq \beta_t \leq \beta_{t+1}(u)$, then

$$-k\frac{(\beta_{t+1}(u) - \beta_t)\hat{u} - (\beta_t - \beta_{t+1}(d))\hat{d}}{\hat{u} - \hat{d}};$$

- if $\beta_t \geq \beta_{t+1}(u)$, then

$$k\frac{(\beta_{t+1}(u)-\beta_t)\hat{u}-(\beta_{t+1}(d)-\beta_t)\hat{d}}{\hat{u}-\hat{d}}.$$

According to equation (B.2), the corresponding values of g' are thus given by:

$$g'(\beta_t) = \begin{cases} 1-k(-\hat{u}+\hat{d})/(\hat{u}-\hat{d}) = 1+k, \\ 1+k(\hat{u}+\hat{d})/(\hat{u}-\hat{d}), \\ 1-k(\hat{u}-\hat{d})/(\hat{u}-\hat{d}) = 1-k. \end{cases} \tag{B.3}$$

Since

$$-k \leq k \leq k(\hat{u}+\hat{d})/(\hat{u}-\hat{d}),$$

we have $g' \geq 1-k$ which is positive according to Assertion 2.

Moreover, $\lim_{x\to\pm\infty} g(x) = \pm\infty$. Similarly to the case $\beta_{t+1}(u) < \beta_{t+1}(d)$, g has a unique zero β_t.

Solving for α_t by substituting in (3.3) or (3.4), we get the corresponding unique solution (α_t, β_t).

Since g is monotone increasing:

$$\begin{array}{ll} g(\beta_{t+1}(u)) \geq 0 \Rightarrow \beta_t \leq \beta_{t+1}(u), & f(\beta_{t+1}(u)) < 0 \Rightarrow \beta_t > \beta_{t+1}(u), \\ g(\beta_{t+1}(d)) \geq 0 \Rightarrow \beta_t \leq \beta_{t+1}(d), & f(\beta_{t+1}(d) < 0 \Rightarrow \beta > \beta_{t+1}(d). \end{array}$$

As a consequence, in order to compute the solution of (3.3) and (3.4) for $t = 1, \ldots, T$, we calculate $g(\beta_{t+1}(u))$ and $g(\beta_{t+1}(d))$ and deduce the expression of $g(\beta_t)$ as the following:

$$|\beta_{t+1}(u)-\beta_t| = \begin{cases} \beta_{t+1}(u)-\beta_t \text{ if } g(\beta_{t+1}(u)) \geq 0, \\ \beta_t-\beta_{t+1}(u) \text{ if } g(\beta_{t+1}(u)) < 0. \end{cases}$$

and

$$|\beta_{t+1}(d)-\beta_t| = \begin{cases} \beta_{t+1}(d)-\beta_t \text{ if } g(\beta_{t+1}(d)) \geq 0, \\ \beta_t-\beta_{t+1}(d) \text{ if } g(\beta_{t+1}(d)) < 0. \end{cases}$$

Equation (B.1) is thus transformed into a linear equation which can be readily solved. □

Remark B.0.3. Boyle-Vorst's algorithm for the long call position is equivalent to the second case of Palmer's theorem. However, for the short call position, Boyle and Vorst added the stronger conditions:

$$\hat{u}(1-k) \geq 1+k, \tag{B.4}$$

$$\hat{d}(1+k) \geq 1-k, \tag{B.5}$$

and the condition that none of the terminal price lies in $[K/(1+k), K(1-k)]$.
Equations (B.4) and (B.5) (implying $k(\hat{u}+\hat{d})(\hat{u}-\hat{d}) < 1$) guaranties the unicity of β_t, whereas the final condition of Boyle-Vorst's theorem ensures that $\beta_{t+1}(u) \leq \beta_t \leq \beta_{t+1}(d)$ if $\beta_{t+1}(u) \leq \beta_{t+1}(d)$,

which is used for approximation results. Nevertheless, β_t may in fact not lie between $\beta_{t+1}(u)$ and $\beta_{t+1}(d)$.

Therefore, Palmer's theorem works for both long and short call positions with no additional conditions for the latter case. Moreover, it also works with any kind of settlement, whereas Boyle-Vorst's algorithm only works for settlements with physical delivery of the underlying asset.

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