LLM Geometry, Fermions, Gauge Theory/Gravity Correspondence

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May 24, 2007

Abstract

In this project, we provide a simple overview of gauge symmetry or local symmetry, in order to introduce LLM geometries, which we will explore in some detail. In particular, the connection between regularity of the supergravity solution and the emergence of the phase space of free fermions will be demonstrated. For a few concrete examples, explicit examples of LLM geometries will be constructed and the geodesics in these geometries will be obtained.

Resumé

Dans ce projet, nous donnons une simple aperçue de symmetrie de gauge ou symmetrie locale, dans le but d'introduire la géométrie LLM, qui sera explorée en details. En particulier, la connection entre la regularité de la solution supergravité et l'emergence de l'espace de phase de fermions libres sera demontrée. Dans quelques cas concrets, des exemples de géométries LLM seront construites et pour ces géométries les géodésiques seront obtenues.

Contents

1. Introduction

String theory is a model of fundamental physics whose building blocks are one-dimensional extended objects called strings, as opposed to the zero-dimensional point particles that form the basis for the Standard Model of particle physics. String Theory is often used as shorthand for Superstring theory, as well as related theories such as M-theory. String theorists are attempting to adjust the Standard Model by removing the assumption in quantum mechanics that particles are point-like. By removing this assumption and replacing the point-like particles with strings, it appears that a sensible quantum theory of gravity naturally emerges. Moreover, string theory may be able to "unify" the known natural interactions (gravitational, electromagnetic, weak nuclear and strong nuclear) by describing them with the same set of equations.

There are different versions of string theory, depending on factors such as whether or not supersymmetry is incorporated into the formulation. These versions are thought to be related to each other as different limits of one theory, coined M-theory. However, there is a huge number of possible solutions to string theory as it is currently understood. Thus it has been claimed by some scientists that string theory may not be falsifiable and may have no predictive power.

Studies of string theory have revealed that it predicts higher-dimensional objects called branes. String theory strongly suggests the existence of ten or eleven spacetime dimensions, as opposed to the usual four (three spatial and one temporal) used in relativity theory; however the theory can describe universes with four effective (observable) spacetime dimensions by a variety of methods.

Interest in string theory is driven largely by the hope that it will prove to be a consistent theory of quantum gravity or even a "theory of everything". It can also naturally describe interactions similar to electromagnetism and the other forces of nature. Superstring theories include fermions, the building blocks of matter, and incorporate supersymmetry, a conjectured (but unobserved) symmetry of nature relating bosons and fermions. It is not yet known whether string theory will be able to describe a universe with the precise collection of forces and particles that is observed, nor how much freedom the theory allows to choose those details. String theory is thought to be a certain limit of another, more fundamental theory, M-theory, which is only partly defined and is not well understood.

String theory is formulated in terms of an action principle, either the Nambu-Goto action or the Polyakov action [dMK05], which describes how strings move through space and time. Like springs with no external force applied, the strings tend to shrink, thus minimizing their potential energy, but conservation of energy prevents them from disappearing, and instead they oscillate. By applying the ideas of quantum mechanics to strings it is possible to deduce the different vibrational modes of strings, and that each vibrational state appearing to be a different particle. The mass of each particle, and the fashion with which it can interact, are determined by the way the string vibrates. The string can vibrate in many different modes, just like a guitar string is able to produce many different notes. The different modes, each corresponding to a different particle states, make up the "spectrum" of the theory. Strings can split and combine, which would appear at large scales like particles emitting and absorbing other particles, presumably giving rise to the known interactions between particles.

String theory includes both open strings, which have two distinct endpoints, and closed strings, where the endpoints are joined to make a complete loop. The two types of string behave in slightly different ways, yielding two different spectra. For example, in most string theories, one of the closed string modes is the graviton, and one of the open string modes is the photon. Because the two ends of an open string can always meet and connect, forming a closed string, there are no string theories without closed strings.

The earliest string model, the bosonic string, which incorporated only bosons, describes at low energies a quantum gravity theory, which also includes (if open strings are incorporated as well) gauge fields such as the photon (or, more generally, any Yang-Mills theory). However, this model has problems. Most importantly, the theory has a fundamental instability, believed to result in the decay (at least partially) of space-time itself. As the name implies, the spectrum of particles contains only bosons, particles which, like the photon, obey particular rules of behavior. Roughly speaking, bosons are the constituents of radiation, but not of matter, which is made of fermions. Investigating how a string theory may include fermions in its spectrum led to the invention of supersymmetry, a mathematical relation between bosons and fermions. String theories which include fermionic vibrations are now known as superstring theories; several different kinds have been described, but all are now thought to be different limits of M-theory. The theory of general relativity combine with notions of supersymmetry is called supergravity.

There are five theories of superstring, we have Type I has closed and open strings, no tachyon; type IIA has closed strings, no tachyon, fermions; and type IIB has closed strings, fermions no tachyon, and two "heterotic" string theories $(SO(32) \text{ and } E_8 \times E_8)$. The five superstring theories are connected to one another as particular limit of M-theory.

By studying black hole thermodynamics, 't Hooft and Susskind conjectured that any region in spacetime has a maximum entropy proportional of the area of the boundary of that region and not proportional to the volume of that region as one might have suspected. This suggests that it may be possible to formulate quantum gravity as a theory without gravity that lives on the boundary of the space inside which you want to describe quantum gravity. This is called the holographic principle because it says that quantum gravity is simply a hologram, it looks (d+1) dimensional but is is only d dimensional.

A concrete example of the holographic principle is provided by the AdS/CFT correspondence. This correspondence says more than the holographic principle, because it claims a relation between a very specific theory of quantum gravity (type IIB superstring theory on the $AdS_5 \times S^5$ background) and a very specific field theory (maximally supersymmetric N=4 super Yang-Mills theory). When we say that "there is a relation" we can't be too precise. To define the link precisely remains an open research problem. We suspect that for every question you can ask in quantum gravity, there is a corresponding question to be resolve in the field theory.

This is progress, but it is only a conjecture. We don't know how to prove the above correspondence. What makes the proof difficult is that when the field theory is weakly coupled, so that we can solve it, the dual gravity geometry is highly curved and we must include quantum gravity corrections. We don't know how to do that. When the geometry has only small curvature, the field theory is strongly coupled, and we do not have the tools for that. The best chance of making progress is to find quantities that can be computed at weak coupling, but which should give the same answer at strong coupling. The only way this can happen is if there is a symmetry which protected the state from getting corrections. The 1/2 BPS states are states of this type. They are invariant under 1/2 the supersymmetries of the theory and they don't get corrected. A supersymmetry transformation is any transformation that will turn a boson into a fermion, or more generally, mix fermion and bosons.

In this essay we are not considering string theory, or black hole thermodynamics, or the holographic principle or the AdS/CFT correspondence or Yang-Mills theory. These are beyond the scope of the present work. In [HIM04], 1/2 BPS geometries are constructed. I will focus my study on LLM geometries (Hai Lin, Oleg Lunin, and Juan Maldacena). The 1/2 BPS geometries preserve half of supersymmetries of type IIB string theory. LLM geometries are specific examples of 1/2 BPS geometries. The link to free fermions is seen because the boundary condition looks like a free fermion occupation number map of phase space.

This essay is made up of two chapter. The first chapter concerns gauge symmetry. In section 2.1 we discuss the concepts of symmetry and dynamics, which play a very important role in physics nowadays. I will also specify exactly the meaning of the word "symmetry" in physics. Later on we will see how this notion of symmetry is related to the concept of dynamics. The overall concept of gauge theory is not straight forward to understand. With some examples in sections 2.2,2.3 and 2.4 we illustrate gauge theory in Quantum mechanics, electromagnetics and quantum electrodynamics. The purpose of doing so is to deepen our understanding. We finish chapter one by saying some thing about the relationship between gauge theory and geometry.

The chapter two contains the most important part of our work and the central aim is LLM construction. In the section 2.5 we start with the notion of geodesics and killing vectors. The idea behind LLM construction is the notions of symmetries. That is why we dedicate one chapter to its study and try to understand the notion of symmetry and gauge. A geodesics is the shortest path between two given points. In the plane, geodesics are segments of lines; and in the sphere geodesics are parallel and the meridian or the big circle. Killing vectors are the generator of isometry, or symmetry in the case of the metric. We show that the standard metric has six killing vectors, three translations and three rotations. The three rotations are exactly the infinitesimal generator of the group $O(3)$. In section 3.1 We review the general structure of LLM solution. The required geometry has an $SO(4) \times SO(4) \times \mathbb{R}$ isometry. This leads us to the metric 3.19 [HIM04]. Requiring certain symmetry of space-time, Line, Lunin and Maldacena were able to construct the solution from 3.22 to 3.29. This study use the metric 3.19 which is require in the LLM geometry and the solution define by 3.30 or 3.34. The section 3.2 regards the regularity of the LLM solution. In the plane $y = 0$ the LLM geometry is singular until the solution specify by a certain function in the plane $x_1 - x_2$ takes a special value. We will show the connection between the LLM solution and the occupation number of fermions. In the section 3.3, we find the solution explicitly, given the boundary conditions obtained in section 3.2. The section 3.5 gives some examples where we write the metric and find the geodesics either analytically or numerically. We end by a short conclusion.

2. Gauge Symmetry

2.1 Symmetry and dynamics

Symmetry plays a central role in physics, not only as a means to solve otherwise analytically intractable problems but also as important tool in providing profound insight into properties of interactions and conservation laws. Indeed when it concerns a continuous symmetry associated to the the notion of a Lie group, Noether‘s theorem for example establishes a direct relation between the existence of such continuous symmetries and of conserve quantities, the so called Noether‘s charges. Futhermore within the Hamiltonian formulation the algebra of Poisson brackets proves to be identical to the abstract algebra of the Lie group symmetry of which the charges are the generators. In other words the abstract algebra of the Lie symmetries group is then realised on the phase space of the system through the conserved quantities and their Poisson brackets.

When canonical quantisation is able to proceed in a manner consistent with the Lie algebra of symmetries, the associated quantum operators then generate, as linear transformations, the symmetry algebra on the associated quantum state state space . It is here that the whole of Lie algebra representation theory becomes most relevant, with particular a classification of the possible quantum representation of a given classical symmetry. As an example, if a system is invariant under $SO(3)$ rotations in three Euclidian space dimensions, the Noether charges correspond to the total angular-momentum vector of that system, whose Poisson brackets are isomorphic to the Lie algebra $so(3)$ of that Lie group. When quantised, one is then able to classify the quantum space of states in terms of representations of $so(3)$, namely, since the algebra of $SO(3)$ and $SU(2)$ are identical. A system may also be invariant under a discrete symmetry of which the group elements depend on a collection of parameters taking only a discrete set of values [Gov91].

In this chapter we shall limit our discussion to continuous symmetries and we focus the section for ther by excluding Noether theorems from our deliberation. Rather than delving into gauge theory in details we will describe some simple examples from electromagnetism and quantum electrodynamics in order to understand it. Gauge symmetry, or local symmetry, is a continuous Lie symmetry of which the continuous parameters themselves may be continuous functions of time or even spacetime in a field theory context. As such, when one takes values for these function parameters , one has what is also often called a global symmetry. This was the situation address earlier, corresponding to Nother‘s first theorem, that leading to conserved Noether charges. Now let specify what is exactly meant by symmetry. Clearly it must consist in a a transformation of the configuration space variables, q^n, of the system, possibly in combination with a transformation in the time variable,t, of the form,

$$t^{'} = t^{'}(t), \quad q^{n^{'}}(t^{'}) = q^{n^{'}}(q^n, t). \tag{2.1}$$

In the case of discrete symmetry the functions $t^{'}(t)$ and $q^{n^{'}}(q^n, t)$ would depend on a collection of parameters taking values in a discrete set. In the case of a continuous symmetry forming a Lie group, the symmetry parameters take their values in a continuous set. For instance, the two or three dimensional spherically symmetric harmonic oscillator is invariant under all space rotations. For the Lie group $SO(3)$ or $SU(2)$ their elements may be characterised in terms of angular variables, the rotation angle in the plane in the first case, or the three Euler angles in the second case. Each of these angular variable takes values in continuous finite intervals. All these descriptions concern transformation, when does a transformation qualify symmetry?

In the context of physics and the dynamics of systems, what one means by the concept of symmetry is not that a particular configuration of the system is left invariant by the transformation, but rather that the space of solutions to its equations of motions is left invariant under the symmetry transformation. A symmetry is a transformation which maps any given solution to an equation of motion into another solution to the same equations of motion. Often one say that a symmetry leaves the form of the equation of motion invariant. This means that when expressing the equations of motions in terms of the as yet untransformed variables or the transformed ones, the functional relation between these variables in each case is identical, this means that they have the same form [Gov91].

Given the above representation of such a transformation, when expressed in terms of the variable carrying a prime, $t^{'}$ and $q^{n^{'}}$, the equations they obey are the same as those for the variables not carrying that prime, namely t and q^n.
How then does the action of a system transform under a symmetry? Since the equation of motion are form invariant, it is clear that under a symmetry the action may only change by a total time derivative. Indeed as is well knows, action that differs only by a total time derivative share identical equations of motion. Hence when transformations for the above class define a symmetry of dynamics, the action of the system must transform according to

$$S\left[q^{n^{'}}\right] = \int dt^{'} L\left(q^{n^{'}}, \frac{dq^n}{dt}\right) = \int dt\left[L\left(q^n, \frac{dq^n}{dt}\right) + \frac{d \wedge (q^n, t)}{dt}\right] \tag{2.2}$$

where $\wedge(q^n, t)$ is some function implicitly defined through the transformation of the action under the symmetry. In particular, this function depends on the parameters of the symmetry group, say the rotation angles in the case of rotational symmetry in space. Using the composition law for differentials

$$dt^{'} = dt\frac{dt^{'}}{dt}, \tag{2.3}$$

a system is invariant under a symmetry transformation if its Lagrange function changes according to

$$\frac{dt^{'}}{dt} L\left(q^{n^{'}}, \frac{dt}{dt^{'}}\frac{dq^n}{dt}\right) = L\left(q^n, \frac{dq^n}{dt}\right) + \frac{d \wedge (q^n, t)}{dt} \tag{2.4}$$

In the case of field theory where the action is given as the spacetime integral of the lagrangian density L, the above total time derivative must be replaced by a total surface term. The simplest gauge theory is electromagnetism. And by far the simplest way to present electromagnetism as a gauge theory is through the non-relativistic Schrodinger equation of a particle moving in empty space.

2.2 Gauge invariance in quantum mechanics

The electric and magnetic fields can be described in terms of $A^\mu = (\phi, \overrightarrow{A})$

$$\overrightarrow{E} = -\overrightarrow{\nabla}\phi - \frac{\partial \overrightarrow{A}}{\partial t}; \quad \overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A},$$

which are invariant under the gauge transformation

$$\phi \longrightarrow \phi' = \phi - \frac{\partial \chi(x,t)}{\partial t}, \quad \overrightarrow{A} \longrightarrow \overrightarrow{A}' = \overrightarrow{A} + \overrightarrow{\nabla} \times \chi$$

Lest us consider the Hamiltonian that gives rise to the Lorentz force:

$$H = \frac{1}{2m}\left(\overrightarrow{p} - q\overrightarrow{A}\right)^2 + q\phi.$$

With the prescription $\overrightarrow{p} \longrightarrow -i\overrightarrow{\nabla}$ we obtain the Schrodinger equation for a particle in an electromagnetic field:

$$\left[\frac{1}{2m}\left(-i\overrightarrow{\nabla} - q\overrightarrow{A}\right)^2 + q\phi\right]\psi(x,t) = i\frac{\partial \psi(x,t)}{\partial t}.$$

which can be written as:

$$\frac{1}{2m}\left(-i\overrightarrow{D}\right)^2 \psi = iD_0\psi$$

This is obtained by making the substitution

$$\overrightarrow{\nabla} \longrightarrow \overrightarrow{D} = \overrightarrow{\nabla} - iq\overrightarrow{A}, \quad \frac{\partial}{\partial t} \longrightarrow D_0 = \frac{\partial}{\partial t} + iq\phi$$

on the free Schrodinger equation. Now, if we make the gauge transformation

$$G : (\phi, \overrightarrow{A}) \longrightarrow (\phi', \overrightarrow{A}'),$$

the solution of

$$\frac{1}{2m}(-i\overrightarrow{D}')^2\psi' = iD_0\psi'$$

does not describe the same physics. We need to make a phase transformation on the matter field:

$$\psi' = \exp(iq\chi)\psi$$

with the same $\chi = \chi(x,t)$. The derivative transforms as:

$$\begin{aligned} \overrightarrow{D}'\psi' &= \left[\overrightarrow{\nabla} - iq(\overrightarrow{A} + \overrightarrow{\nabla}\chi)\right]\exp(iq\chi)\psi \\ &= \exp(iq\chi)(\overrightarrow{\nabla}\psi + iq(iq\chi))\psi \\ &- iq\overrightarrow{A}\exp(iq\chi)\psi - iq(\overrightarrow{\nabla}\chi)\exp(iq\chi)\psi \\ &= \exp(iq\chi)\overrightarrow{D}\chi \end{aligned} \tag{2.5}$$

and

$$D_0^{'}\psi^{'} = \exp(iq\chi)D_0\chi,$$

the Schrodinger equation becomes:

$$\begin{aligned}
\frac{1}{2m}(-iD^{'})^2\psi^{'} &= \frac{1}{2m}(-i\overrightarrow{D}^{'})(-i\overrightarrow{D}^{'}\psi^{'}) && (2.6)\\
&= \frac{1}{2m}(-i\overrightarrow{D}^{'})\left[-i\exp\left(-iq\chi\right)\overrightarrow{D}\psi\right]\\
&= \exp(iq\chi)\frac{1}{2m}(-i\overrightarrow{D})^2\psi\\
&= \exp(iq\chi)(iD_0)\psi = iD_0^{'}\psi && (2.7)
\end{aligned}$$

and now both fields describe the same physics since $|\psi|^2 = |\psi^{'}|^2$. In order to make all variables invariants we should substitute:

$$\overrightarrow{\nabla} \longrightarrow \overrightarrow{D}, \quad \frac{\partial}{\partial t} \longrightarrow D_0$$

Now let us consider another example.

2.3 Example from electrodynamics: the extended Landau problem

Let us consider first a non relativistic massive charged particle confined to the Euclidean plane, with position vector $\overrightarrow{r} = (x, y)$, x and y being cartesian coordinates. The particle is subjected to a static homogeneous magnetic field $\overrightarrow{B}$ perpendicular to the plane, a statistic homogeneous electric field $\overrightarrow{E}$ lying within the plane, and a spherically symmetric harmonic force of potential energy $V(x, y) = \frac{1}{2}(x^2 + y^2)$.
It is easy to check that the appropriate choice of gauge, the "symmetric gauge" for the vector potential $\overrightarrow{A}$ and the vector potential associated to the electric field are respectively $\overrightarrow{A}(\overrightarrow{r}) = \frac{1}{2}\overrightarrow{B} \times \overrightarrow{r}$ and $\Phi(\overrightarrow{r}) = -\overrightarrow{r}.\overrightarrow{E}$. This means that they yield to the same electric and magnetic field. Indeed the homogeneous equations of Maxwell are given by:

$$\overrightarrow{\nabla} \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0 \quad \text{and} \quad \overrightarrow{\nabla}.\overrightarrow{B} = 0. \tag{2.8}$$

From these equations in $\mathbb{R}^3$, we can show that the electric and magnetic field are given by;

$$\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \frac{\partial \overrightarrow{A}}{\partial t} \quad \textit{and} \quad \overrightarrow{B} = \overrightarrow{\nabla} \times \overrightarrow{A}. \tag{2.9}$$

For the specification of gauge potentials $\overrightarrow{A}(\overrightarrow{r}) = \frac{1}{2}\overrightarrow{B} \times \overrightarrow{r}$ and $\Phi(\overrightarrow{r}) = -\overrightarrow{r}.\overrightarrow{E}$ we have

$$\Phi(t, \overrightarrow{r}) = -xE_1 - yE_2, \quad A_1(t, \overrightarrow{r}) = -\frac{1}{2}By, \quad A_2(t, \overrightarrow{r}) = +\frac{1}{2}Bx, \quad A_3(t, \overrightarrow{r}) = 0. \tag{2.10}$$

one then has

$$-\overrightarrow{\nabla}\Phi - \partial_t \overrightarrow{A}(t, \overrightarrow{r}) = \overrightarrow{E} \quad and \quad \overrightarrow{\nabla} \times \overrightarrow{A}(t, \overrightarrow{r}) = \overrightarrow{B} \tag{2.11}$$

which correctly reproduces the configuration of the electric and magnetic fields.
We could imagine a general gauge by considering the following transformation for these gauge potentials,

$$\Phi^{'}(t, \overrightarrow{r}) = \Phi(t, \overrightarrow{r}) - \partial_t \psi(t, \overrightarrow{r}), \quad \overrightarrow{A}^{'}(t, \overrightarrow{r}) = \overrightarrow{A}(t, \overrightarrow{r}) + \overrightarrow{\nabla}\psi(t, \overrightarrow{r}) \tag{2.12}$$

where $\psi(t, \overrightarrow{r})$ is an arbitrary function of time and space. Under the conditions of this problem, all choices of gauge potentials reproducing the same configuration of the electric and magnetic fields $\overrightarrow{E}$ and $\overrightarrow{B}$ are related through

$$\Phi^{'}(t, \overrightarrow{r}) = -\overrightarrow{r}.\overrightarrow{E} - \partial_t \psi(t, \overrightarrow{r}), \quad \overrightarrow{A}^{'}(t, \overrightarrow{r}) = \frac{1}{2}\overrightarrow{B} \times \overrightarrow{r} + \overrightarrow{\nabla}\psi(t, \overrightarrow{r}) \tag{2.13}$$

This is the general gauge we were trying to generate. In order to restrict to a particular what are for instance all gauge choices for the electric and magnetic fields which have time independent, or static gauge potentials. In such case, the equations for the gauge parameter function $\psi(t, \overrightarrow{r}))$ are:

$$\partial_t^2 \psi(t, \overrightarrow{r}) = 0, \quad \overrightarrow{\nabla}\partial_t \psi(t, \overrightarrow{r}) = 0. \tag{2.14}$$

Hence

$$\psi(t, \overrightarrow{r}) = \Psi_0 t + \psi_0(\overrightarrow{r}), \tag{2.15}$$

where Ψ_0 is an arbitrary real constant and ψ_0 an arbitrary function of the space coordinates. Then we have the following gauge:

$$\Phi^{'}(t, \overrightarrow{r}) = -\overrightarrow{r}.\overrightarrow{E} - \Psi_0, \quad \overrightarrow{A}^{'} = \frac{1}{2}\overrightarrow{B} \times \overrightarrow{r} + \overrightarrow{\nabla}\psi_0(\overrightarrow{r}) \tag{2.16}$$

2.4 Example from quantum electrodynamics

We now consider a theory where the symmetry transforms are space time dependent, i.e $\varepsilon^a = \varepsilon^a(x)$. They are know as local or gauge symmetries. We shall see that such symmetries can be used to generate dynamics, the gauge interaction. The prototype gauge theory is quantum electrodynamics. It is now believed that all interactions are described by some form of gauge theory [CF90].
Consider the Lagrangian for free electron field $\psi(x)$

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^u \partial_u - m)\psi(x) \tag{2.17}$$

It has a global $U(1)$ symmetry corresponding to the invariance of the theory under a phase change

$$\psi(x) = \psi^{'}(x) = e^{-i\alpha}\psi(x) \tag{2.18}$$

$$\bar{\psi}(x) = \bar{\psi}^{'}(x) = e^{i\alpha}\bar{\psi}(x) \tag{2.19}$$

We are going to turn this symmetry into a local symmetry, i.e to a gauge symmetry, by replacing α with $\alpha(x)$. This local symmetry, we will call abelian gauge symmetry since the group $U(1)$ is abelian. We want to construct a theory which will be invariant under space time dependent phase change:

$$\psi(x) = \psi^{'}(x) = e^{-i\alpha(x)}\psi(x) \tag{2.20}$$

$$\bar{\psi}(x) = \bar{\psi}^{'}(x) = e^{i\alpha(x)}\bar{\psi}(x). \tag{2.21}$$

The derivative term in (2.19) is

$$\bar{\psi}(x)\partial_u\psi(x) = \bar{\psi}^{'}(x)\partial_u\psi^{'}(x) = \bar{\psi}(x)e^{i\alpha(x)}\partial_u(e^{-i\alpha(x)}\psi(x))$$

The second term, $\bar{\psi}(x)m\psi(x)$, of (2.17) spoils the derivative. We need to form a gauge-covariant derivative D_u, to replace ∂_u and $D_u\psi(x)$ the simple transformation

$$D_u\psi(x) = [D_u\psi(x)] = e^{-\alpha(x)}D_u\psi(x) \tag{2.22}$$

so that combination $\bar{\psi}(x)D_u\psi(u)$ is gauge invariant. The action of the covariant derivative on the field will not change the transformation property of the field. We can generalise this by enlarging the theory with a new vector field, $A_u(x)$ the gauge field and for the covariant derivative as:

$$D_u\psi = (\partial_u + ieA_u)\psi. \tag{2.23}$$

Then the transformation law for the covariant derivative (2.22) will be satisfied if the gauge field $A_u(x)$ has the transformation

$$A_u(x) \to A_u^{'}(x) + \frac{1}{e}\partial_u\alpha(x), \tag{2.24}$$

The corresponding lagrangian density is then

$$\mathcal{L} = \bar{\psi}(x)i\gamma^u(\partial_u + ieA_u)\psi(x) - m\bar{\psi}(x)\psi(x). \tag{2.25}$$

To make the gauge field a true dynamical variable, we need to add a term to the Lagrangian involving its derivatives. The simplest invariant term in dimensions four or less is

$$\mathcal{L}_{\mathcal{A}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{2.26}$$

where

$$F_{\mu\nu} = \partial_u A_\nu - \partial_\nu A_\mu$$

$F_{\mu\nu}$ is in fact gauge invariant by itself. Combining (2.26) and (2.25) we obtain the known density lagrangian of quantum electrodynamics. For more details see [CF90]

$$\mathcal{L} = \bar{\psi}(x)i\gamma^u(\partial_u + ieA_u)\psi(x) - m\bar{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.27}$$

The transformations represented by $\exp(i\chi(x,t))$, or $\exp(x)$, that is rotations in the plane, form an abelian group; these transformations are commutative. When the gauge transformation is not commutative, we enter in the realm of Yang Mills theory which we are not equipped to consider in this essay. We finish this chapter with a short discussion of gauge theory and geometry.

2.5 Gauge theory and Geometry

There are fascinating connections between Riemannian geometry, gauge theories, and quantum mechanics. The connections between gauge theory and geometry are mentioned in nearly every textbook on quantum field theory. How this geometry can be visualised is not so often mentioned.

Gauge theories can be expressed using the principal bundle constructions. Principal bundles are defined with respect to a symmetry group, and one can have a distinct representations of the symmetry group act on different objects in the theory. Because of the multiple representations, not every principle bundle can be mapped back into a single vector bundle construction.

In contrast, traditional general relativity is defined around coordinate reparameterization invariance, typically represented on a vector bundle. All vector objects are sections of the same vector bundle. For this reason, the vector bundle associated with the tangent space is relatively easy to understand, especially when thought of in terms of an embedding space. A natural basis for the tangent space is provided by tangent vectors along space-time coordinates. Vectors are geometrical objects on the tangent space that can be expressed in terms of different basis choices, but the vector itself is a basis-independent object. Although one might not be able to think easily in more than three dimensions, Riemannian geometry with the aid of an embedding space provides a visual tool to understand the geometrical significance of the curvature that leading to gravitational force.

What about the geometrical significance of gauge theory. Principle bundles add a great deal of structure beyond the vector bundles at the heart of coordinate reparameterization invariance in general relativity. We use a trivial bundle as a type of embedding space. This trivial bundle allows one to compare vectors at different space-time points. The idea is analogous to embedding a 2-dimensional sphere in 3-dimensional Eucledian space to understand the role of parallel transport in the covariant derivatives of Riemannian geometry. We have mostly studied $U(1)$ gauge theories represented as $SO(2)$ gauge theories. What one would normally think of as the $SO(2)$ gauge fibre is now seen as a two-dimensional real vector bundle inserted within the trivial vector bundle of larger dimension. The wave-function is a gauge-invariant vector field from the embedding space that can be described in terms of any basis that spans the two-dimensional gauge fibre. A gauge transformation changes the basis by which one describes the gauge fibre. The variation of the gauge-fibre at different space-time points manifests itself as electric and magnetic fields.

In summary the essential point is that, in any space where the coordinate are position dependent, the significance of comparing two vectors(or any two tensors) at different points is lost. The standard way of dealing with this problem is to introduce the notion of parallel transport or affine connection. In the case of physical curved space-time the christofell symbol is introduced and in the case of internal charge the gauge fields are introduced. They compasate for the change of local frames at each time point. For more details see [CF90] and [Gua04].

3. LLM Construction

3.1 Geodesics and killing vectors

A very important concept that enters into LLM constructions is the notion of symmetries. We think of a manifold M as possessing a symmetry if the geometry is invariant under a certain transformation that maps M to itself, i.e. the metric is the same one point to another. Different tensor fields may have different symmetries; symmetries of the metric are called isometries. Consider four dimensional Minkowski space

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.1}$$

Translations $x^\mu \longrightarrow x^\mu + a^\mu$ and Lorentz transformations $x^\mu \longrightarrow \Lambda^\mu_\nu x^\nu$ (Λ^μ_ν is the Lorentz transformation matrix) are isometries. Indeed the metric is invariant under translations when the metric is independent of the coordinate functions x^μ. Whenever this is the case, we always have an isometry,

$$\partial_{\sigma *} g_{\mu\nu} = 0, \quad \text{implies} \quad x^{\sigma *} \longrightarrow x^{\sigma *} + a^{\sigma *} \quad is \quad \text{an} \quad \text{isometry.} \tag{3.2}$$

Isometries of this form have an immediate consequence for the motion of test particles as described by the geodesic equation. The geodesic equation can be written in terms of the four momentum $p^\mu = mU^\mu$ as

$$p^\lambda \nabla_\lambda p^\mu = 0. \tag{3.3}$$

We can lower the index μ (the covariant derivative of the metric vanishes) and expand the covariant derivative out to obtain

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\sigma_{\lambda\mu} p^\lambda p_\sigma = 0. \tag{3.4}$$

The first term tells us how the metric changes along the path

$$p^\lambda \partial_\lambda p_\mu = m\frac{dx^\lambda}{d\tau}\partial_\lambda p_\mu = m\frac{dp_\mu}{d\tau}.$$

The second term is

$$\Gamma^\sigma_{\lambda\mu} p^\lambda p^\sigma = \frac{1}{2} g^{\sigma\nu}(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}) p^\lambda p_\sigma = \frac{1}{2}(\partial_\mu g_{\nu\lambda}) p^\lambda p^\nu. \tag{3.5}$$

Thus, the geodesic equation can be written as

$$m\frac{dp_\mu}{d\tau} = \frac{1}{2}(\partial_\mu g_{\nu\lambda}) p^\lambda p^\nu. \tag{3.6}$$

Thus, if the metric is independent of $x^{\sigma*}$, the momentum component $p_{\sigma*}$ is a constant of the motion

$$\partial_{\sigma*} g_{\mu\nu} = 0, \quad \text{implies} \quad \text{that} \quad \frac{dp_{\sigma_*}}{d\tau} = 0 \tag{3.7}$$

Although independence of the metric on one or more coordinates does imply an isometry, the converse is not true. Indeed, we could transform to a complicated coordinate system where the translational symmetries were not obvious. Such a coordinate transformation does change the metric components, but it does not change the underlying geometry, and this is what the symmetry is truly characterising.
Let us study the systematic procedure to identify an isometry.

If $x^{\sigma*}$ is the coordinate that $g_{\mu\nu}$ is independent of, consider the vector

$$K = \partial_{\sigma*}. \tag{3.8}$$

In component notation we have

$$K^{\mu} = (\partial_{\sigma*})^{\mu} = \delta^{\mu}_{\sigma*}. \tag{3.9}$$

K^{μ} generates the isometry: the transformation under which the geometry is invariant is expressed infinitesimally as a motion in the direction of K^{μ}. In terms of the vector, we can write the conserved momentum in a covariant way

$$p_{\sigma*} = K^{\nu} p_{\nu} = K_{\nu} p^{\nu}. \tag{3.10}$$

Constancy of this quantity can be expressed as

$$\frac{dp_{\sigma*}}{d\tau} = 0 \quad \longleftrightarrow p^{\mu} \nabla_{\mu} (K_{\nu} p^{\nu}) = 0. \tag{3.11}$$

After a little manipulation of (3.11) we have

$$p^{\mu} \nabla_{\mu} (K_{\nu} p^{\nu}) = p^{\mu} p^{\nu} \nabla_{\mu} K_{\nu} + p^{\mu} K_{\nu} \nabla_{\mu} p^{v} = p^{\mu} p^{\nu} \nabla_{\mu} K_{\nu}$$

Thus we have K_{ν}, which is a quantity that satisfy an equation of the form:

$$\nabla(_{\mu} K_{\nu}) = 0 \tag{3.12}$$

The equality in (3.12) implies that $K_{\nu} p^{\nu}$ is conserved along the geodesic. This last equation is know as Killing's equation and vector fields that satisfy it are known as Killing vector fields. Vector fields on a manifold are in a one-to-one correspondence with continuous symmetries of the metric on that manifold. Every killing vector implies the existence of conserved quantities associated with geodesic motion. This makes sense: by definition, the metric is not changing along the direction of the Killing vector. So, loosely speaking a free particle will not feel any forces in this direction and hence the component of momentum in this direction is conserved. Let us find, for instance, the killing vector of the standard metric

$$ds^2 = dx^2 + dy^2 + dz^2. \tag{3.13}$$

Clearly the Killing vectors are (in rectangular coordinates)

$$K_x = \partial_x, \quad K_y = \partial_y, \quad K_z = \partial_z \tag{3.14}$$

In cylindrical coordinates (r, ϕ, z) we have $x = r\cos\phi,\ y = r\sin\phi,\ z = z$:

and

$$dx = \cos\phi dr - r\sin\phi d\phi, \quad dy = \sin\phi dr + r\cos\phi d\phi, \quad dz = dz.$$

We can write the metric (3.13) in the following form:

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \tag{3.15}$$

Then $R_z = \partial_\phi$ is a killing vector field. We have in rectangular coordinates,

$$\begin{aligned} R_z &= \partial_\phi \\ &= \partial_{\sigma*}\partial_\phi x^{\sigma*} \\ &= \partial_x\partial_\phi x + \partial_y\partial_\phi y + \partial_z\partial_\phi z. \end{aligned} \tag{3.16}$$

Note that $x^{\sigma*} = x, y, z$ and then

$$R_z = -y\partial_x + x\partial_y = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{3.17}$$

This generates rotation about the z-axis. Similarly,

$$R_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z},\ R_y = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} \tag{3.18}$$

which generate rotations about the x-axis and the y-axis, respectively.

3.2 General structure of the LLM solution

In [HIM04] a class of BPS solutions of type IIB supergravity is constructed. The geometry required has an $SO(4)\times SO(4)\times\mathbb{R}$ isometry. This implies that the geometry will contains two three sphere and a Killing vector. We expect only the five-form field strength to be excited. So we assume the existence of a geometry of the form:

$$ds^2 = g_{\mu\nu}dx_\mu dx^\nu + e^{H+G}d\Omega_3^2 + e^{H-G}d\widetilde{\Omega}_3^2 \tag{3.19}$$

$$F_5 = F_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\Omega_3 + \widetilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge d\widetilde{\Omega}_3^2. \tag{3.20}$$

$\mu,\nu = 0,1..,3$

$$F = e^{3G} *_3 \widetilde{F},\ F = dB,\ \widetilde{F} = d\widetilde{B} \tag{3.21}$$

We can define a function $z = z(x_1, x_2, y)$ which determines the entire solution (up to choice of gauge) that we will discuss below. By requiring the background spacetime to have certain symmetries, Line, Lunin and Maldacena were able to construct the following solution [HIM04]:

$$\begin{aligned}
ds^2 &= -h^{-2}(dt+V_idx^i)^2 + h^2(dy^2+dx^idx^i) + ye^{G}d\Omega_3^2 + ye^{-G}d\widetilde{\Omega}_3^2 && (3.22)\\
h^{-2} &= 2y\cosh G && (3.23)\\
y\partial_y V_i &= \epsilon_{ij}\partial_i z,\ y(\partial_i V_j - \partial_j V_i) = \epsilon_{ij}\partial_y z && (3.24)\\
z &= \frac{1}{2}\tanh G && (3.25)\\
F &= dB_t \wedge (dt+V) + B_t dV + d\widetilde{B} && (3.26)\\
\widetilde{F} &= d\widetilde{B}_t \wedge (dt+V) + \widetilde{B}_t dV + d\hat{\widetilde{B}} && (3.27)\\
B_t &= -\frac{1}{4}y^2e^{2G},\ \widetilde{B}_t = -\frac{1}{4}y^2e^{-2G} && (3.28)\\
d\hat{B} &= -\frac{1}{4}y^3 *_3 d(\frac{z+\frac{1}{2}}{y^2}),\ d\hat{\widetilde{B}} = -\frac{1}{4}y^3 *_3 d(\frac{z+\frac{1}{2}}{y^2}) && (3.29)
\end{aligned}$$

with $y \geq 0$.

where $i = 1,2$ and $*_3$ is the flat epsilon symbol in three dimensions parameterised by y, x_1, x_2. Although both the derivation of these equations and their physical relevance are extremely interesting, they are beyond the scope of this essay. In this essay I will simply discuss in detail the boundary conditions that one must apply to obtain a regular geometry and I will construct some examples of LLM spacetimes. Given these solutions, I will describe the geodesics that arise from the resulting geometries. From (3.24) we see that the full solutions is determined in terms of a single function z. This function obeys the linear differential equation,

$$\partial_i\partial_i z + y\partial_y(\frac{\partial_y z}{y}) = 0, \tag{3.30}$$

or

$$(\partial_1^2 + \partial_2^2 + \partial_y^2 - \frac{1}{y}\partial_y)z = 0, \tag{3.31}$$

3.3 Regularity of the LLM solution

Given a solution to the supergravity equation of motion, a very important question to ask is if there are any singularities in the geometry. We want to consider solutions that are free of singularities. Singularities in a given geometry usually arise when a sphere, in the geometry shrinks to zero size, or a circle shrinks to zero size or the volume of a torus in the geometry goes to zero: something is shrinking, which implies that the manifold is getting more and more curved. In the limit that the shrinking object reaches zero size, the curvature blows up to infinity and a singularity is produced. In order to have a clear idea of this phenomenon, let us consider for instance the surface of the cone defined by the parametric equation below,

$$r(\varphi,\rho) = (\rho\sin\alpha\cos\varphi, \rho\sin\alpha\sin\varphi, \rho\cos\alpha).$$

From the elementary theory of a surface in $\mathbb{R}^3$, the metric of this surface is given by

$$ds^2 = Ed\varphi^2 + 2Fd\varphi d\alpha + Gd\alpha^2,$$

where

$$E =< r_\varphi, r_\varphi >= \rho^2\sin^2\alpha,\ F =< r_\varphi, r_\rho >= 0,\ G =< r_\rho, r_\rho >= 1,$$

and $r_\varphi = \frac{\partial r}{\partial\varphi}$, $r_\rho = \frac{\partial r}{\partial\rho}$ then

$$ds^2 = \rho^2\sin^2\alpha d\varphi^2 + d\rho^2.$$

The normal vector to the surface of the cone is defined by:

$$\overrightarrow{N} = r_\varphi \wedge r_\rho = (\frac{1}{2}\rho\sin 2\alpha\cos\varphi, \frac{1}{2}\rho\sin 2\alpha\sin\varphi, -\rho\sin^2\alpha),$$

and the unit normal vector is

$$\overrightarrow{n} = \frac{\overrightarrow{N}}{\|\overrightarrow{N}\|} = r_\varphi \wedge r_\rho = (\cos\alpha\cos\varphi, \cos\alpha\sin\varphi, -\sin^2\alpha),$$

The second fundamental form is defined by

$$ds^2 = Ld\varphi^2 + 2Md\varphi d\alpha + Nd\alpha^2,$$

where,

$$L =< r_{\varphi\varphi}, \overrightarrow{n} >= -\frac{1}{2}\sin 2\alpha,\ M =< r_{\varphi\alpha}, \overrightarrow{n} >= 0,\ N =< r_{\rho\rho}, \overrightarrow{n} >= 0,$$

The curvature of the metric of the cone is a solution of the equation;

$$(L-\mu E)(N-\mu G) - (M-\mu F)(M-\mu F) = 0,$$

We find μ=0 (Gauss curvature) and $\mu = \frac{1}{\rho}cotg(\alpha) \sim \frac{\alpha}{\rho}$ for small α. When $\rho = 0$, which corresponds to the tip of the cone, the curvature goes to infinity. Hence the geometry defined by the metric of the cone presents a singularity located at the tip of the cone.

In the LLM geometry we use the metric (3.22). There are two spheres. One has radius squared $r^2 = ye^G$ and the other $\widetilde{r}^2 = ye^{-G}$, thus, $r^2\widetilde{r}^2 = y^2$. It is obvious that the $y = 0$ plane with coordinates (x_1, x_2) is a potential location of singularities until z has special values [Buc04]. On this plane, we have $r^2\widetilde{r}^2 = 0$. This is the only place where the radii vanish, as the exponential can not have a zero. When $z = +\frac{1}{2}$, and from (3.25), we obtain

$$\frac{1}{2} = \frac{1}{2}\tanh(G),\ so\ G \text{ goes } to\ +\infty$$

Let $z(x_1, x_2, y = 0) = z_0$, and expanding z near $y = 0$ we have:

$$z(x_1, x_2, y) = z_0 + \frac{\partial z}{\partial y}(y = 0)y + \frac{\partial^2 z}{\partial y^2}(y = 0)y^2 + ...$$

From (3.23) and (3.25) one has:

$$z = \frac{1}{2}\sqrt{1 - 4h^4y^2},\ \text{and}\ \frac{\partial z}{\partial y}(y = 0) = -\frac{2h^{4y}}{\frac{1}{2}\sqrt{1 - 4h^4y2}}(y = 0) = 0.$$

We can show that $\frac{\partial^2 z}{\partial y^2}(y = 0) \neq 0$, and setting $\frac{\partial^2 z}{\partial y^2}(y = 0) = f(x)$ to be a certain function of $x = (x_1, x_2)$, we can write the expansion of z in the following form:

$$z = z_0 + y^2 f(x) + ...$$

on the other hand and we also have

$$z = \frac{1}{2} = \frac{1}{2}\frac{e^G - e^G}{e^G + e^{-G}} \sim \frac{1}{2}(1 - 2e^{-2G} + ...),$$

Hence

$$z \sim \frac{1}{2} - e^{-2G} = \frac{1}{2} + y^2 f(x) + ...,\ f(x) < 0,$$

and then

$$e^{-G} \sim yc(x),\ \text{where}\quad c(x) = \sqrt{-f(x)},$$

And from (3.23)

$$h^2 = \frac{1}{2y}sechG = \frac{2}{e^G + e^{-G}} \sim \frac{1}{y}e^{-G} \sim c(x)\ \text{when}\ G \to \infty$$

The metric in the y direction and the second 3-sphere direction becomes:

$$ds^2 = h^2dy^2 + ye^{-G}d\widetilde{\Omega}_3^2 \sim c(x)(dy^2 + d\widetilde{\Omega}_3^2), \tag{3.32}$$

which describes the boundary of space time in this case. Here h is finite and the radius of the first 3-sphere also remains finite. Doing the calculation, we have for $z = \frac{-1}{2}$

$$\frac{-1}{2} = \frac{1}{2}\tanh(G),\ so\ \text{G goes to } -\infty$$

$$z = -\frac{1}{2} = -\frac{1}{2}\frac{e^G - e^G}{e^G + e^{-G}} \sim \frac{1}{2}(1 - 2e^{2G} + ...)$$

$$z \sim -\frac{1}{2} - e^{-2G} = -\frac{1}{2} + y^2 f(x) + ...,\ f(x) > 0$$

$$e^G \sim yc(x),\ \text{where } c(x) = \sqrt{f(x)},\ and\ h \sim c(x)$$

and

$$ds^2 = ye^G d\Omega_3^2 + h^2 dy^2 \sim c(x)(dy^2 + d\Omega_3^2), \tag{3.33}$$

So we have the same boundary metric as before. We can summarise the above:

since the product of the radii of the spheres vanishes on the plane $y = 0$, singularities are avoided only if $z(x_1, x_2, y = 0)$ takes the values $\pm\frac{1}{2}$, meaning that one or the other sphere remains. A solution is then fully specified by the data of the $x_1 - x_2$ plane on which $z(x_1, x_2, y = 0)$ takes each value, which [HIM04] color-code as black$=-\frac{1}{2}$, where Ω_3 shrinks, and white$=+\frac{1}{2}$ where $\widetilde{\Omega}_3$ shrinks. The final result is that those LLM geometries which are specified by black and white configurations on the (x_1, x_2) plane with $z = \frac{1}{2}$ or $z = -\frac{1}{2}$ have a boundary of the form $\mathbb{R} \times S^3$. The connection to free fermions is seen because of the boundary condition that we apply. We can not put more than one fermion into a single state. Thus, if we plot the fermion phase space, it can be colored with two colors corresponding to "occupied" or "empty".

3.4 Solution for the given boundary conditions

Following [MO05], the solution is well defined for z restricted to the range $-\frac{1}{2} \leq z \leq \frac{1}{2}$ as seen before. The equation (3.31) implies that z takes its maximum and minimum on the boundary of its domain of definitions $\mathcal{D} \in \mathbb{R}^2 \times \mathbb{R}^+$. In order to determine the solution to the supergravity equation, we need to specify a choice of regions $\mathcal{D}$ and define a function z_0 on $\partial\mathcal{D}$ such that:

$$z = z_0\ on\ \partial\mathcal{D} \quad -1/2 \leqslant z_0 \leqslant 1/2$$

Let us define $\Phi = \frac{z}{y^2}$. Equation for z becomes the Laplace equation in six dimensional space $\mathbb{R}^2 \times \mathbb{R}^4$ for Φ where x_1, x_2 are the coordinates on the $\mathbb{R}^2$ with spherical symmetry in four of the dimensions. The radial variable in these four dimensions is then y. With (3.30) and the definition of Φ both singular for $y = 0$, the boundary value of z on the $y = 0$ plane gives the charge sources for this equation in six dimension. Thus with $\Phi = \frac{z}{y^2}$ it is easy to see that Φ satisfies the equation:

$$(\partial_1^2\Phi + \partial_2^2\Phi + \frac{1}{y^3}\partial_y(y^3\partial_y\Phi)) = *_6 d *_6 d\Phi = \Delta\Phi = -4\pi^2 z_0 \delta^4(y)\chi(\mathcal{D}) \tag{3.34}$$

where $\delta^4(y)$ is the delta function of the variable y and $\chi(\mathcal{D})$ is the characteristic function of $\mathcal{D}$. Let us say some thing to indicate the origin of the last equality of (3.34).

In four dimensional space we have,

$$\int_{S^3\subset R^4} \triangle(\frac{z}{y^2})d^4V = \int_{S^3\subset R^4} \triangledown^2(\frac{z}{y^2})d^4V = \int_{S^3\subset R^4} \triangledown(\triangledown\frac{z}{y^2})d^4V.$$

The hyperdimensional volume of the space which a $(n-1)$-sphere encloses, the n-ball, is:

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}. \tag{3.35}$$

The "surface area" of this $(n-1)$-sphere is

$$S_n = \frac{dV_n}{dR} = \frac{nV_n}{R} = \frac{2\pi^{\frac{n}{2}}R^{n-1}}{\Gamma(\frac{n}{2})}, \tag{3.36}$$

By the divergence theorem we have the following equality

$$\int_{S^3\subset R^4} \triangledown(\triangledown\frac{z}{y^2})d^4\overrightarrow{r} = \int_{\partial S^3} \triangledown(\frac{z}{y^2})\overrightarrow{n}\,dS = \int_{\partial S^3} \partial(\frac{z_0}{y^2})\overrightarrow{n}\,dS = \int_{\partial R^4} 2(\frac{z_0}{y^3})\overrightarrow{n}\,dS = -\frac{4\pi^2R^3}{y^3}z_0$$

where in the last integral we have performed the surface integral using (3.36). The integration is over a small sphere of radius R in four dimensions. Now for $y > 0$ and $R \to 0$, the integral becomes zero, similarly, for $y = R$ and $R \to 0$ the integral becomes $-4\pi^2$. Therefore,

$$\nabla^2(\frac{z}{y^2}) = -4\pi^2 z_0\delta^4(y). \tag{3.37}$$

Let us now find the six dimensional Green function. The fundamental solution F of (3.37) is a solution of:

$$\nabla^2 F(\overrightarrow{r}, \overrightarrow{r_0}) = \delta^6(|\overrightarrow{r} - \overrightarrow{r_0}|) \tag{3.38}$$

Integrating over all space in six dimension we get:

$$\int_{\mathbb{R}^6} \nabla^2 F(\overrightarrow{r}, \overrightarrow{r_0})d^6V = \int_{\mathbb{R}^6} \delta^6(|\overrightarrow{r} - \overrightarrow{r_0}|)d^6V = 1 \tag{3.39}$$

Using green's theorem on the left hand side we have

$$\int_{\mathcal{D}\subset\mathbb{R}^6} \nabla^2 F(\overrightarrow{r}, \overrightarrow{r_0})d^6V = \int_{\mathcal{D}\subset\mathbb{R}^6} \triangledown(\triangledown F)d^6V = \int_{\partial\mathcal{D}} \triangledown F\overrightarrow{n}\,d^6S = \int_{\partial\mathcal{D}} \frac{dF}{d\overrightarrow{r}}\overrightarrow{n}\,d^6S = 1 \tag{3.40}$$

Using (3.36) and performing the surface integral in the last equality we get,

$$\frac{dF}{dr}2\pi r^5\Big|_{r=R} = 1$$

One has

$$F(r) = -\frac{1}{4\pi^3 r^4} + c$$

where c is such that $F(\overrightarrow{r}, \overrightarrow{r_0}) = 0$ when $r \to \infty$ and setting $\overrightarrow{r} = \overrightarrow{X}$, $\overrightarrow{r_0} = \overrightarrow{X'}$ we have

$$F(|\overrightarrow{x} - \overrightarrow{x}'|) = -\frac{1}{4\pi^3 |\overrightarrow{x} - \overrightarrow{x}'|^4},$$

which is our green's function. If we choose $\mathcal{D} \subset \mathbb{R}^2 \times \mathbb{R}^+$, from (3.37) we have,

$$\Phi = \int_D -4\pi^2 z_0 F(|\overrightarrow{x} - \overrightarrow{x}'|) dx_1' dx_2',$$

Hence

$$z(x_1, x_2, y) = y^2 \Phi = y^2 \frac{1}{\pi} \int_D \frac{z(x_1', x_2', 0) dx_1' dx_2'}{\left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2\right]^2} + \sigma \tag{3.41}$$

Where σ is the contribution from infinity which arises in the case that z is constant outside of very large radius. And using (3.24) we have:

$$V_i(x_1, x_2, 0) = \frac{\varepsilon_{ij}}{\pi} \int_D \frac{z(x_1', x_2', 0)(x_j - x_j') dx_1' dx_2'}{\left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2\right]^2}, \tag{3.42}$$

By Stokes theorem we can also write

$$z(x_1, x_2, y) = -\frac{1}{2\pi} \int_{\partial D} dl\, n_i' \frac{x_i - x_i'}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2} + \sigma. \tag{3.43}$$

Indeed

$$\begin{aligned} z(x_1, x_2, y) &= \int_D \nabla \left(-\frac{1}{2\pi} \frac{z(x_1', x_2', 0)(x_i - x_i')}{\left[(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2\right]}\right) dx_1' dx_2' \\ &= -\frac{1}{2\pi} \int_{\partial D} dl\, n_i' \frac{x_i - x_i'}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2} \end{aligned}$$

and for V,

$$V_i(x_1, x_2, y) = \frac{\varepsilon_{ij}}{2\pi} \int_{\partial D} dl\, \frac{dx_j'}{(x_1 - x_1')^2 + (x_2 - x_2')^2 + y^2} \tag{3.44}$$

3.5 Some examples

3.5.1 Example 1

Let us now consider a simple solution which is associated to the half filled plane. We have the boundary conditions

$$z(x_1', x_2', 0) = \frac{1}{2} sign\, x_2', \tag{3.45}$$

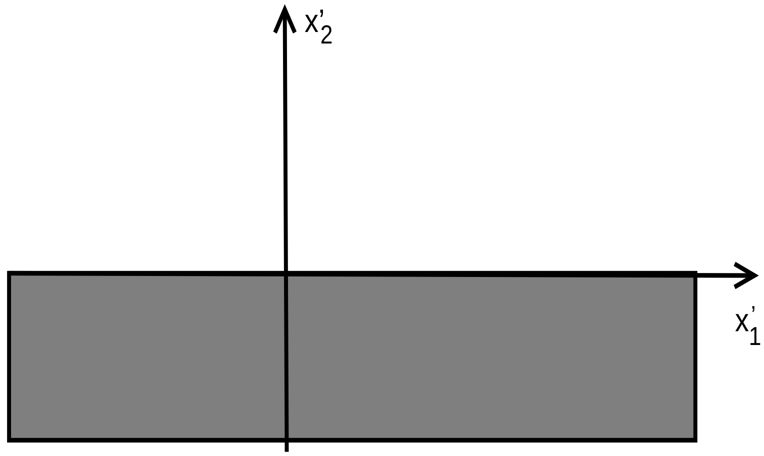

Figure 3.1: The half filled plane

The integral we need to compute is (3.43) of LLM, $\sigma = 0$ no charge at infinity. The path we integrate over is the boundary of the droplet, i.e. the axis, so that $x_2^{'} = 0$. The unit vector $n_i^{'}$ points toward the $z = \frac{1}{2}$ region,i.e. $\overrightarrow{n} = (0, -1, 0)$. Thus using (3.43) we obtain

$$z(x_1, x_2, y) = \frac{x_2}{2\pi}\int_{-\infty}^{+\infty}\frac{dx_1^{'}}{(x_1 - x_1^{'})^2 + x_2^2 + y^2} = \frac{x_2}{2\pi}\frac{1}{\sqrt{x_2^2 + y^2}}\arctan\frac{x_1 - x_1^{'}}{\sqrt{x_2^2 + y^2}}\Big|_{-\infty}^{+\infty} = \frac{1}{2}\frac{x_2}{\sqrt{x_2^2 + y^2}},$$

Using relation (3.44) it is easy to demonstrate that $V_2 = 0$ $(x_2^{'} = 0)$ and

$$V_1 = \frac{1}{2\pi} = \int_{-\infty}^{+\infty}\frac{dx_1^{'}}{(x_1 - x_1^{'})^2 + x_2^2 + y^2} = \frac{1}{2\pi}\frac{1}{\sqrt{x_2^2 + y^2}}arctg\frac{x_1 - x_1^{'}}{\sqrt{x_2^2 + y^2}}\Big|_{-\infty}^{+\infty} = \frac{1}{2}\frac{1}{\sqrt{x_2^2 + y^2}},$$

Performing a change in variable,

$$y = r_1 r_2, \quad x_2 = \frac{1}{2}(r_1^2 - r_2^2) \tag{3.46}$$

and using (3.25) and the expression for z, we have $\tanh G = \frac{x_2}{\sqrt{x_2^2+y^2}}$. Moreover from (3.23) $h^2 = (2y)^{-1} sechG = \frac{1}{2\sqrt{x_2^2+y^2}} = V_1$ and $h^{-2} = \frac{1}{V_1}$
We have also from (3.46)

$$dy = r_1 dr_2 + r_2 dr_1, \quad dx = r_1 dr_1 - r_2 dr_2, \tag{3.47}$$

We can write the metric (3.22) in the following form

$$\begin{aligned} ds^2 &= -h^{-2}dt^2 - 2dtdx + h^2(dy^2 + dx^2) \\ &= -2dtdx_1 - (r_1^2 + r_2^2)dt^2 + dr_1^2 + dr_2^2 \end{aligned} \tag{3.48}$$

The final solution is smooth instead of the fact that on the $y = 0$ plane V diverges at the boundary between two regions ($x_2 = 0$). In fact this computation shows that, in general, the boundary between two regions is smooth. The reason, is that, locally, the boundary between two regions looks like the plane wave, and therefore, we will generate a non singular metric. Let us consider the metric(3.48) and find the geodesics.

$$x^\mu = (t, x_1, r_1, r_2)\ \mu = 1, 2, 3, 4, \tag{3.49}$$

$$x^1 = t, x^2 = x_1, x^3 = r_1, x^4 = r_2, \tag{3.50}$$

The matrix notation of (3.48) is

$$g = \begin{pmatrix} -r_1^2 - r_2^2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so $g_{11} = -r_1^2 - r_2^2$, $g_{12} = g_{21} = -1$, $g_{33} = 1$,$g_{44} = 1$. Using equation (3.6) of geodesics, we see:

$$\begin{cases} \frac{dp_1}{d\tau} = \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial x^1} p^\lambda p^\mu = 0 \\ \frac{dp_2}{d\tau} = \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial x^2} p^\lambda p^\mu = 0 \\ \frac{dp_3}{d\tau} = \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial x^2} p^\lambda p^\mu = -r_1 p^1.p^1 \\ \frac{dp_4}{d\tau} = \frac{1}{2}\frac{\partial g_{\nu\lambda}}{\partial x^4} p^\lambda p^\mu = -r_2 p^1.p^1 \end{cases} \tag{3.51}$$

As $p_\mu = mU_\mu$ and $\frac{dp_\mu}{d\tau} = m\frac{dU_\mu}{d\tau}$ the set of equations (3.51) becomes

$$\begin{cases} \dfrac{d^2t}{d\tau^2} = 0 \\ \dfrac{d^2x_1}{d\tau^2} = 0 \end{cases} \quad \text{and} \quad \begin{matrix} \dfrac{d^2r_1}{d\tau^2} = -r_1(\dfrac{dt}{d\tau})^2 \\ \dfrac{d^2r_2}{d\tau^2} = -r_2(\dfrac{dt}{d\tau})^2 \end{matrix}$$

Using (3.50), we can write the set of equations as (3.5.1)

$$\begin{cases} \ddot{x}_1 = 0 \\ \ddot{x}_2 = 0 \\ \ddot{x}_3 = -x_3(\dot{x}_1)^2 \\ \ddot{x}_4 = -x_4(\dot{x}_1)^2 \end{cases} \tag{3.52}$$

We can solve the system (3.52) analytically, and we obtain,

$$\dot{x}_1 = a_1, \quad x_1(\tau) = a_1\tau + a_2, \quad and \quad \dot{x}_2 = b_1, \quad x_2(\tau) = b_1\tau + b_2 \tag{3.53}$$

Plugging (3.53) into last two equations of (3.52) we find

$$\ddot{x}_3 = -x_3 a_1^2, \quad x_3(\tau) = A_1\cos(a_1\tau) + A_2\sin(a_1\tau) \tag{3.54}$$

and

$$\ddot{x}_4 = -x_4 a_1^2, \quad x_4(\tau) = B_1\cos(a_1\tau) + B_2\sin(a_1\tau) \tag{3.55}$$

Given the initial conditions

$$x_1(0) = 1, \quad \dot{x}_1(0) = 2, \quad x_2(0) = 1, \quad \dot{x}_2(0) = 1$$
$$x_3(0) = 1.5, \quad \dot{x}_3(0) = 1, \quad x_4(0) = 1, \quad \dot{x}_0(4) = 1$$

we can write the solutions as

$$x_1 = 2\tau + 1, \quad x_2 = \tau + 1,$$
$$x_3 = 1.5\cos(\tau) + \sin(\tau), \quad x_4 = \cos(\tau) + \sin(\tau)$$

We can solve the system (3.52) numerically, either with Python or Matlab. For this purpose, let us reduce the system (3.52) to the first order by setting,

$$y_1 = x_1,\ y_2 = \dot{x}_1,\ y_3 = x_2,\ y_4 = \dot{x}_2,\ y_5 = x_3,\ y_6 = \dot{x}_3,\ y_7 = x_4,\ y_8 = \dot{x}_4$$

The system (3.52) becomes,

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = 0 \\ \dot{y}_3 = y_4 \\ \dot{y}_4 = 0 \\ \dot{y}_5 = y_6 \\ \dot{y}_6 = -y_5 * (y_2)^2 \\ \dot{y}_7 = y_8 \\ \dot{y}_8 = -y_7 * (y_2)^2 \end{cases} \tag{3.56}$$

The corresponding python code is given overleaf:

```
import scipy
def geodesic(w,t):
y1,y2,y3,y4,y5,y6,y7,y8=w
return scipy.array([y2,0,y4,0,y6,-y5*(y2)**2,y8,-y7*(y2)**2])
w0=scipy.array([1.,2.,1.,1.,1.5,1.,1.,1.])
timepoints=scipy.arange(0.,6.28,.02)
trajectory=scipy.integrate.odeint(geodesic,w0,timepoints)
scipy.gplt.plot(trajectory[:,0:8:2])
scipy.gplt.title('plot')
scipy.gplt.xtitle('absciss')
scipy.gplt.ytitle('ordinate')
scipy.gplt.output("lois.eps", 'postscript')
```

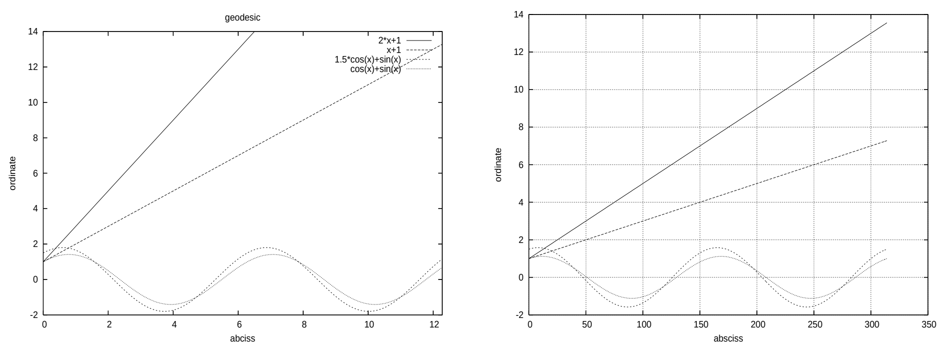

Figure 3.2: geodesic, left analytic solution and right numerical solution with python

3.5.2 Example 2

As another example, let us introduce a function $\widetilde{z} = z - \frac{1}{2}$ [HIM04]. The Laplace equation for $\frac{\widetilde{z}}{y^2}$ has sources on a disk of radius r_0. Choosing polar coordinates r,ϕ in the x_1, x_2 plane, We obtain

$$x_1 = r\cos\phi, \quad dx_1 = dr\cos\phi - r\sin\phi dr \tag{3.57}$$
$$x_2 = r\sin\phi, \quad dx_2 = dr\sin\phi + r\cos\phi dr \tag{3.58}$$

$$Vidx^i = V_\phi d\phi, \tag{3.59}$$

$$\widetilde{z}(r,y) = -\frac{y^2}{\pi}\int_{Disk}\frac{r^{'}dr^{'}d\phi^{'}}{[r^2 + r^2 - 2rr^{'}\cos\phi + y^2]^2}, \tag{3.60}$$

Integrating with respect to $r^{'}$ from 0 to r_0, and with respect to ϕ from 0 to 2π we find

$$\widetilde{z}(r, y; r_0) = \frac{r^2 - r_0^2 + y^2}{2\sqrt{r^2 + r_0^2 + y^2 - 4r^2 r_0^2}} - \frac{1}{2}, \tag{3.61}$$

and

$$V_\phi = -r \sin\phi V_1 + r\cos\phi V_2 = -\frac{1}{\pi}\int_{\partial\mathcal{D}} \frac{rr' \cos\phi' d\phi'}{r^2 + r^2 - 2rr'\cos\phi + y^2}, \tag{3.62}$$

Integrating again with respect to ϕ' from 0 to 2π and setting $r' = r_0$ we have

$$V_\phi(r, y; r_0) = -\frac{1}{2}\Big(\frac{r^2 - r_0^2 + y^2}{\sqrt{r^2 + r_0^2 + y^2 - 4r^2 r_0^2}} - 1\Big), \tag{3.63}$$

Performing the change of coordinates

$$y = r_0 \sinh\rho \sin\theta \tag{3.64}$$
$$r = r_0 \cosh\rho\cos\theta \tag{3.65}$$
$$\widetilde{\phi} = \phi - t \tag{3.66}$$

we can write (3.61) and (3.63) as

$$z = \frac{-\sin^2\theta}{\cosh^2\rho - \cos^2\theta}, \quad V_\phi = \frac{-\cos^2\theta}{\cosh^2\rho - \cos^2\theta}, \tag{3.67}$$

We have

$$\begin{aligned}
ye^G &= yshG + y\cosh G \\
2ye^G &= 2y\sinh G + 2y\cosh G \\
&= 2y\sinh G + \frac{1}{h^2} \quad (from (3.23)) \\
&= 2y\sqrt{\cosh^2 G - 1} + \frac{1}{h^2} \\
2ye^G &= \frac{1}{h^2}\tanh G + \frac{1}{h^2} \\
ye^G &= \frac{1}{2h^2}(2z+1) \\
&= \frac{y(2z+1)}{\sqrt{1-4z^2}} = y\sqrt{\frac{1+2z}{1-2z}} = r_0 \sinh^2\rho
\end{aligned} \tag{3.68}$$

consequently,

$$\begin{aligned}
ye^{-G} &= \frac{1}{2h^2}(1-2z) \\
&= \frac{y(1-2z)}{\sqrt{1-4z^2}} = y\sqrt{\frac{1-2z}{1+2z}} = r_0\sin^2\theta,
\end{aligned} \tag{3.69}$$

From (3.23), (3.25), (3.68), (3.69) we have,

$$\begin{aligned} h^{-2} &= \frac{2y}{\sqrt{1-4z^2}} \\ &= y\sqrt{\frac{1+2z}{1-2z}} + y\sqrt{\frac{1-2z}{1+2z}} \\ &= r_0(\sinh^2\rho + \sin^2\theta) \qquad (3.70) \\ h^2 &= \frac{1}{r_0(\sinh^2\rho + \sin^2\theta)} \qquad (3.71) \end{aligned}$$

Plugging (3.57), (3.58), (3.67), (3.68), (3.69), (3.70) and (3.71) in (3.22), we obtain the metric, usually called the standard metric of $Ads_5 \times S^5$

$$ds^2 = r_0\left(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho d\Omega_3^2 + \cos^2\theta d\widetilde{\phi}^2 \sin^2\theta d\widetilde{\Omega}_3^2\right), \qquad (3.72)$$

$x^\mu = (t, \rho, \Omega_3, \theta, \widetilde{\phi}, \widetilde{\Omega}_3), \quad \mu = 1, 2, 3, 4, 5, 6$
$g_{tt} = -r_0\cosh^2\rho,\ g_{\rho\rho} = 1,\ g_{\Omega_3\Omega_3} = \sinh^2\rho,\ g_{\theta\theta} = 1,\ g_{\widetilde{\phi}\widetilde{\phi}} = \cos^2\theta,\ g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} = \sin^2\theta$

The geodesic equations are

$$(I)\begin{cases} \ddot{t} = 0 \\ \ddot{\rho} = -\frac{1}{2}r_0\sinh 2\rho(\dot{t}^2 - \dot{\Omega}_3^{\ 2}) \\ \ddot{\Omega}_3 = 0 \\ \ddot{\theta} = -\frac{1}{2}r_0\sin 2\theta(\dot{\widetilde{\phi}}^2 - \dot{\widetilde{\Omega}}_3^{\ 2}) \\ \ddot{\widetilde{\phi}} = 0 \\ \ddot{\widetilde{\Omega}}_3 = 0 \end{cases}$$

We can also solve this set of equations above numerically. Set $t = x_1, \dot{t} = x_2, \rho = x_3, \dot{\rho} = x_4, \Omega_3 = x_5, \dot{\Omega}_3 = x_6, \theta = x_7, \dot{\theta} = x_8, \widetilde{\phi} = x_9, \dot{\widetilde{\phi}} = x_{10}, \widetilde{\Omega}_3 = x_{11}, \dot{\widetilde{\Omega}}_3 = x_{12}, r_0 = 1$. The set of equations (I) becomes,

$$(II)\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = 0\\ \dot{x}_3 = x_4\\ \dot{x}_4 = -\frac{1}{2}r_0\sinh 2x_3(x_2^2 - x_6^2)\\ \dot{x}_5 = x_6\\ \dot{x}_6 = 0\\ \dot{x}_7 = x_8\\ \dot{x}_8 = -\frac{1}{2}r_0\sin 2x_7(x_{10}^2 - x_{12}^2)\\ \dot{x}_9 = x_{10}\\ \dot{x}_{10} = 0\\ \dot{x}_{11} = x_{12}\\ \dot{x}_{12} = 0\end{cases}$$

Let us set $r_0 = 1$. The corresponding code is given below.

```
import scipy
def geodesic(w,t):
y1,y2,y3,y4,y5,y6,y7,y8,y9,y10,y11,y12=w
return scipy.array([y2,0,y4,-.5*scipy.sinh(2*y3*(y2**2-y6**2)),y6,0,y8,
-.5*scipy.sin(2*y7*(y10**2-y12**2)),y10,0,y12,0])
w0=scipy.array([1.2,1.5,1.,1.,1.5,1.,1.,1.,1.,1.5,1.,1.])
timepoints=scipy.arange(0.,6.28,.02)
trajectory=scipy.integrate.odeint(geodesic,w0,timepoints)
#print trajectory
scipy.gplt.plot(trajectory[:,0:12:2])
#scipy.gplt.title('plot')
scipy.gplt.xtitle('absciss')
scipy.gplt.ytitle('ordinate')
scipy.gplt.output("lois3.eps", 'postscript')
#scipy.gplt('set output "lois.eps"')
```

As before, for given initial condition, we can solve the set of differential equations of order one with python and generate the graph (see figure 3.3).

3.5.3 Example 3

We can construct the solution that corresponds to a superposition of (3.61) and (3.63). In this case, of concentric rings, the metric is given by

$$ds^2 = -h^{-2}(dt + V_\phi d\phi)^2 + h^2(dy^2 + r^2 d\phi^2) + ye^G d\Omega_3^2 + ye^{-G} d\widetilde{\Omega}_3^2$$

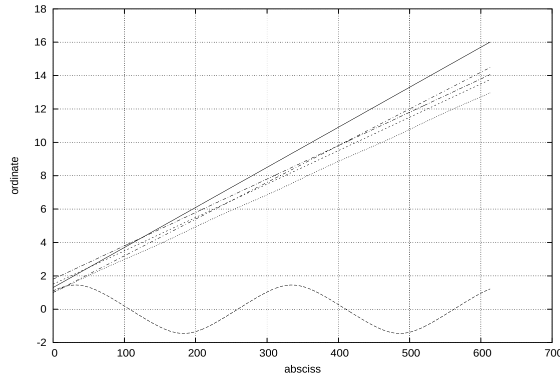

Figure 3.3: numerical solutions; geodesics

or

$$ds^2 = -h^{-2}dt^2 - 2V_\phi dt d\phi + (-h^{-2}V_\phi^2 + r^2)d\phi^2 + h^2 dy^2 + h^2 dr^2 + ye^G d\Omega_3^2 + ye^{-G}d\widetilde{\Omega}_3^2$$

where $x^\mu = (t, \phi, y, r, \Omega_3, \widetilde{\Omega}_3)$.

$$\begin{aligned}
h^{-2} &= 2y\sqrt{1-4\widetilde{z}^2} \\
V_\phi &= \frac{1}{2}\sum_{n=1}^{N}(-1)^n\Big(\frac{r^2+r_n^2+y^2}{\sqrt{(r^2+r_n^2+y^2)-4r^2r_n^2}}-1\Big) \\
\widetilde{z} &= \frac{1}{2}\sum_{n=1}^{N}(-1)^{n+1}\Big(\frac{r^2-r_n^2+y^2}{\sqrt{\big(r^2+r_n^2+y^2\big)-4r^2r_n^2}}-1\Big) \\
ye^G &= y\sqrt{\frac{1+2z}{1-2z}} \\
ye^{-G} &= y\sqrt{\frac{1-2z}{1+2z}} \\
g_{tt} &= -h^{-2} \\
g_{t\phi} &= -V_\phi h^{-2} \\
g_{\phi\phi} &= (-h^{-2}V_\phi^2 + r^2) \\
g_{yy} &= h^2 \\
g_{rr} &= h^2 \\
g_{\Omega_3\Omega_3} &= ye^G \\
g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} &= ye^{-G}
\end{aligned}$$

Where N is the number of rings and r_1 is the radius of the outermost circles, r_2 the next one, etc. we need to have,

$$
\begin{aligned}
\partial_y g_{tt} &= -2\sqrt{1-4\widetilde{z}^2} + 8z(1-4\widetilde{z}^2)^{-1/2}\partial_y \widetilde{z} \\
\partial_r g_{tt} &= 8zy(1-4\widetilde{z}^2)^{-1/2}\partial_r \widetilde{z} \\
\partial_y g_{t\phi} &= -\partial_y(V_\phi)h^{-2} - V_\phi \partial_y h^{-2} \\
\partial_r g_{t\phi} &= -\partial_r(V_\phi)h^{-2} - V_\phi \partial_r h^{-2} \\
\partial_y g_{\phi\phi} &= \partial_y g_{t\phi} \\
\partial_r g_{\phi\phi} &= -\partial_r h^{-2} V_\phi^2 - 2V_\phi \partial V_\phi h^{-2} + 2r \\
\partial_y g_{yy} &= (-(1-4\widetilde{z}) + 4zy\partial_y z)/2y^2(1-4\widetilde{z}^2)^{3/2} \\
\partial_r g_{yy} &= 2/y(1-4\widetilde{z})^{3/2}\partial_r z \\
\partial_y g_{rr} &= \partial_y g_{yy} \\
\partial_r g_{rr} &= \partial_r g_{yy} \\
\partial_y g_{\Omega_3\Omega_3} &= (2/(1-2z^2))(((1+2\widetilde{z})/(1-2\widetilde{z})))^{-1/2}\partial_y \widetilde{z} \\
\partial_r g_{\Omega_3\Omega_3} &= (-4z/(1-2z^2))(((1+2\widetilde{z})/(1-2\widetilde{z})))^{-1/2}\partial_r \widetilde{z} \\
\partial_y g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} &= (-4\widetilde{z}/(1+2z^2))(((1-2\widetilde{z})/(1+2\widetilde{z})))^{-1/2}\partial_y \widetilde{z} \\
\partial_r g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} &= (-2/(1+2z^2))(((1-2\widetilde{z})/(1-2\widetilde{z})))^{-1/2}\partial_r \widetilde{z} \\
\partial_y \widetilde{z} &= \sum_{n=1}^{N}(-1)^{n+1}\frac{y((r^2+r_n^2+y^2)^2-4r^2r_n^2) - y(r^2-r_n^2+y^2)(r^2+r_n^2+y^2)}{(r^2+r_n^2+y^2)^2-4r^2r_n^2)^{3/2}} \\
\partial_r \widetilde{z} &= \sum_{n=1}^{N}(-1)^{n+1}\frac{r((r^2+r_n^2+y^2)^2-4r^2r_n^2) - (r^2-r_n^2+y^2)(r(r^2+r_n^2+y^2)-4rr_n^2)}{(r^2+r_n^2+y^2)^2-4r^2r_n^2)^{3/2}} \\
\partial_y V_\phi &= \sum_{n=1}^{N}(-1)^{n}\frac{y((r^2+r_n^2+y^2)^2-4r^2r_n^2) - y(r^2+r_n^2+y^2)(r^2+r_n^2+y^2)}{(r^2+r_n^2+y^2)^2-4r^2r_n^2)^{3/2}} \\
\partial_r V_\phi &= \sum_{n=1}^{N}(-1)^{n}\frac{r(r(r^2+r_n^2+y^2)^2-4r^2r_n^2) - (r^2+r_n^2+y^2)(r(r^2+r_n^2+y^2)-4rr_n^2)}{(r^2+r_n^2+y^2)^2-4r^2r_n^2)^{3/2}}
\end{aligned}
$$

The geodesic equations are

$$
\begin{cases}
\ddot{t} = 0 \\
\ddot{\phi} = 0 \\
\ddot{y} = \frac{1}{2}\partial_y g_{tt}\dot{t}^2 + \frac{1}{2}\partial_y g_{\phi\phi}\dot{\phi}^2 + \frac{1}{2}\partial_y g_{yy}\dot{y}^2 + \frac{1}{2}\partial_y g_{rr}\dot{r}^2 + \frac{1}{2}\partial_y g_{\Omega_3\Omega_3}\dot{\Omega_3}^2 + \frac{1}{2}\partial_y g_{\widetilde{\Omega}_3\widetilde{\Omega}_3}\dot{\widetilde{\Omega}_3}^2 + \partial_y g_{t\phi}\dot{t}\dot{\phi} \\
\ddot{r} = \frac{1}{2}\partial_r g_{tt}\dot{t}^2 + \frac{1}{2}\partial_r g_{\phi\phi}\dot{\phi}^2 + \frac{1}{2}\partial_r g_{yy}\dot{y}^2 + \frac{1}{2}\partial_r g_{rr}\dot{r}^2 + \frac{1}{2}\partial_r g_{\Omega_3\Omega_3}\dot{\Omega_3}^2 + \frac{1}{2}\partial_r g_{\widetilde{\Omega}_3\widetilde{\Omega}_3}\dot{\widetilde{\Omega}_3}^2 + \partial_r g_{t\phi}\dot{t}\dot{\phi} \\
\ddot{\Omega}_3 = 0 \\
\ddot{\widetilde{\Omega}}_3 = 0
\end{cases}
$$

setting $t = x_1, \dot{t} = x_2, \phi = x_3, \dot{\phi} = x_4, y = x_5, \dot{y} = x_6, r = x_7, \dot{r} = x_8, \Omega_3 = x_9, \dot{\Omega}_3 = x_{10}, \widetilde{\Omega}_3 = x_{11}, \dot{\widetilde{\Omega}_3} = x_{12}$, we have at the first order

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = 0 \\ \dot{x}_5 = x_6 \\ \dot{x}_6 = \frac{1}{2}\partial_y g_{tt} x_2^2 + \frac{1}{2}\partial_y g_{\phi\phi} x_4^2 + \frac{1}{2}\partial_y g_{yy} x_6^2 + \frac{1}{2}\partial_y g_{rr} x_8^2 + \frac{1}{2}\partial_y g_{\Omega_3\Omega_3} x_{10}^2 + \frac{1}{2}\partial_y g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} x_{12}^2 + \partial_y g_{t\phi} x_2 x_4 \\ \dot{x}_7 = x_8 \\ \dot{x}_8 = \frac{1}{2}\partial_r g_{tt} x_2^2 + \frac{1}{2}\partial_r g_{\phi\phi} x_4^2 + \frac{1}{2}\partial_r g_{yy} x_6^2 + \frac{1}{2}\partial_r g_{rr} x_8^2 + \frac{1}{2}\partial_r g_{\Omega_3\Omega_3} x_{10}^2 + \frac{1}{2}\partial_r g_{\widetilde{\Omega}_3\widetilde{\Omega}_3} x_{12}^2 + \partial_r g_{t\phi} x_2 x_4 \\ \dot{x}_9 = x_{10} \\ \dot{x}_{10} = 0 \\ \dot{x}_{11} = x_{12} \\ \dot{x}_{12} = 0 \end{cases} \tag{3.73}$$

It is not simple in general to solve the set of equations (3.73) analitycally. We can also solve this set of differential equations numerically given initial values. See the code in the appendix. The code produce the following plot,

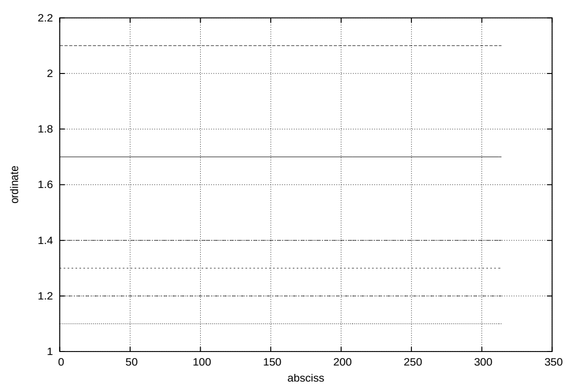

Figure 3.4: numericall solutions, geodesics

A simple particular analytic solution is given by, $x_1 = c_0\tau + c_1$, $x_2 = c_0$, $x_3 = c_2\tau + c_3$, $x_4 = c_2$, if $x_6 = c_4$ and $x_8 = c_5$ then $x_5 = c_4\tau + c_6$ and $x_7 = c_5\tau + c_7$, $x_{10} = c_8$, $x_9 = c_8\tau + c_9$, $x_{12} = c_{10}$, $x_{11} = c_{10}\tau + c_{11}$.

4. Conclusion

The principal aim of this essay was to provide a brief review of LLM geometry. For this purpose we introduced the notion of gauge symmetry, as an illustration of the central role symmetry plays in physics. Through simple examples, we gained an in depth understanding of the notion of gauge without omitting to mention the relationship between gauge theory and geometry. The most important concept underlying the LLM construction is the notion of symmetries. Symmetric transformations leave the metric invariant, and lead us to the concepts of geodesics and the killing vector.

LLM geometry was originally generated to provide solutions to type IIB supergravity [HIM04]. The entire solution can be determined by a function, say $z = z(x_1, x_2, y)$. This function obeys a partial differential equation in the three dimensions y, x_1 and x_2. This geometry is, however not regular in the plane $x_1 - x_2$ (y=0), since the two spheres which are present in the geometry shrink to zero size. When the function $z = z(x_1, x_2, y)$ takes certain special values in the (y=0) plane we obtain a regular geometry, where either one or the other of the two spheres shrinks in a smooth fashion. The corresponding values of z are $\frac{1}{2}$ or $-\frac{1}{2}$. Therefore, a generic LLM solution can be specified by a black (white) color-coding attributed to $z = -1/2(+1/2)$ regions on the $x_1 - x_2$ plane . These boundary conditions of the function z in the (y=0) plane correspond exactly to the occupation number of fermions in the field theory. This means the regularity of the supergravity solution of LLM geometry possesses some sort of connection with the phase space of free fermions.

It would be interesting if we could reproduce the geodesics found in the last part of this essay using only the theory of free fermions. This was not, however, possible given the short time frame of this project.

Appendix A. Quick Review on Continuous Group

A.1 Characterisation

The elements of a continuous groups are characterised by a set of elements which varies continuously over a certain interval. Let r represent the number of continuous parameters. If this number is finite, the continuous group is said to be finite and r is the oder of the continuous group[Jos88].

For instance the set of all real number is a continuous groups of order 1 since any real number can be characterised by one parameter, say x, taking values on the interval $[-\infty, \infty]$. The set of all linear homogeneous transformations of n variables form, a continuous group, of n^2 parmateres is known as the linear group in n dimensions and denoted by $GL(n)$. This group is isomorphic to the group of all non singular matrices of the order n under multiplication. The set of all rotations about an axis is a continuous group of order 1, whose parameter can be chose to be the angle of rotation, say θ, taking values on the interval $[-\pi, \pi]$ or $[-2\pi, 2\pi]$. This group is denoted by $SO(2)$.

A.2 Topological group

A topological group is a group in which both the law of composition and the law of inversion are continuous in all the group elements. ie $G \times G \longrightarrow G : (g, h) \longrightarrow gh \; and \; G \longrightarrow G : g \longrightarrow g^{-1}$ are continuous.

A group is said to be connected if there exists a path connecting any two group elements, or in other words, if its parameter space is connected. The group of rotations about an axis is a connected group, as is the group of proper rotations in three dimension. However, a Continuous group does not necessary to be connected, an important example of which is the rotation inversion group[Jos88].

A topological group is said to be connected if its parameter space is a compact space, that is a closed and a bound space.

A.3 Lie group

A Lie group is a set G which is at the same time a differentiable manifold and group such that its group structure is differentiable. This means that: $G \times G \longrightarrow G : (g, h) \longrightarrow gh \; and \; G \longrightarrow G : g \longrightarrow g^{-1}$ are differentiable.

The set G inherits from its manifold structure properties like dimensionality, compactness, connectedness etc. and from its group structure adjectives like abelian, simple, nilpotent. A Lie subgroup is a subgroup which is also a manifold.

Generator of a Lie group.

The generator of a continuous (Lie) group is introduced by considering elements infinitesimally near to the identity element. In the present case, for small ϕ, the expansion of $R(\phi)$ is of the form[Jon91]:

$$R(\phi) = 1 - i\phi X + O(\phi^2). \tag{A.1}$$

or equivalently

$$-iX = \frac{dR(\phi)}{d\phi}. \tag{A.2}$$

Where R is the element of the representation of the Lie group and X is called the infinitesamal generator of the Lie group.

A.4 The group $SO(2)$

This is the group of proper rotations in 2 dimensions. These are all about the same axis, which we can choose to be the z axis, and are labelled by the angle ϕ with $0 \leq \phi \leq 2\pi$ A typical element of this group can be written as

$$R(\phi) = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix},$$

The generator is therefore

$$X = i\frac{dR(\phi)}{d\phi} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

this is of course one of the Pauli matrices commonly denoted by σ_j. Any 2×2 orthogonal matrix with determinant $+1$ can be written as

$$R(\phi) = \begin{pmatrix} cos\phi & sin\phi \\ -sin\phi & cos\phi \end{pmatrix} = \exp(i\phi\sigma_j)$$

X is hermitian. Indeed this is a consequence of the unitary nature of R. Thus

$$I = R^{\perp}(\phi)R(\phi) = (1 + i\phi X^{\perp})(1 - i\phi X) = 1 - i\phi(X - X^{\perp}) + \mathcal{O}(\varphi^2)$$

Hence $X = X^{\perp}$. In a similar fashion the tracelessness of X derive from the fact that R is a special orthogonal martix with unit determinant.

Let $f \equiv f(x,y)$ and let the operator $T(\phi)$ stand for an orthogonal transformation of the coordinate system. The action of this operator is given by:

$$T(\phi)f(x,y) = f(xcos\phi + ysin\phi, -xsin\phi + ycos\phi).$$

The generator is given by:

$$Xf(x,y) = -i(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})f(x,y),$$

Hence

$$X = -L_x/\hbar$$

Where L_x is the angular component of the angular momentum operator normal to the plane (x,y). Similarly we have

$$L_z f = -i\hbar(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})f(x,y) = (xp_y - yp_x)f = -i\hbar\frac{\partial}{\partial \phi}f \quad \text{(A.3)}$$

An orthogonal transformation of the coordinates in the two dimensional plane (x,y) is then give by:

$$T(\phi) = \exp(-i\phi L_z/\hbar).$$

A.5 The group $SO(3)$

The group of all orthogonal transformations in a three dimensional real vector space is the group $O(3)$. The matrices of this group are divided into two sets. One contains the matrices with determinant $+1$ (which corresponds to a pure rotation) denoted $SO(3)$, and the other contains the matrices with determinant -1 (which corresponds to improper rotation). The generator of $SO(3)$ can be obtained by considering an infinitesimal rotation through an angle ϕ about an axis u. The group of rotation $R_u(\phi)$ for $0 \leq \phi \leq 2\pi$ which is a subgroup of $SO(3)$, is isomorphic to $SO(2)$, hence, as in (A.3) we obtaine $I_u = -L_u/\hbar$, where L_u is the component of the angular momentum operator $\overrightarrow{L}$ along $\overrightarrow{u}$[Jos88]. Since any rotation can be expressed as the product of the three rotations about the cartesian coordinates axes, we see that we need three operators

$$I_x = (-i\phi L_x/\hbar), I_y = (-i\phi L_y/\hbar), I_z = (-i\phi L_z/\hbar),$$

Any operator can then be written as

$$R(\phi) = \exp(-i\phi\overrightarrow{L}.\overrightarrow{u})/\hbar).$$

The full rotation-inversion group $O(3)$ has four parameter $(\alpha, \beta, \gamma, \eta)$ where (α, β, γ) are the parameter of $SO(3)$ and η denotes the determinant of an element and can take values ± 1. The parameter space of $O(3)$ thus consists of two disconnected regions. It is then a four parameter group, three of which are continuous. It is a continuous compact Lie group which is however not connected.

A.6 The group $O(n)$

The set of all matrices of dimension n is a group. This group is denoted by $O(n)$ and is a continuous compact, Lie group, which is however not connected. If x_i form an orthogonal basis, the transformation of $O(n)$ leaves the quadratic form $\sum_{i=1}^{n} x_i^2$ invariant. The subgroup containing proper rotations denoted $SO(n)$, is a connected Lie group with $n(n-1)/2$ parameters $O(n)$ has one discrete parameter in addition to the $n(n-1)/2$ continuous parameter of $SO(n)$.

For example, $O(4)$ is the group of all orthogonal transformations which leaves the quadratic form $x^2 + y^2 + z^2 + u^2$ invariant. If we regard x, y, z, u as the cartesian coordinates axes in a four dimensional Euclidian space, the six parameter of $SO(4)$ can be though of as representing rotations in the six coordinates planes. From our discussion of $SO(2)$ and $SO(3)$, it can be seen that the six generator of $SO(4)$ can be conveniently chosen to be:

$$A_1 = -i(y\partial z - z\partial y), A_2 = -i(z\partial x - x\partial z), A_3 = -i(x\partial y - z\partial x),$$

$$B_1 = -i(x\partial u - u\partial x), B_2 = -i(y\partial u - u\partial y), B_3 = -i(z\partial u - u\partial z).$$

The commutators of these generators with each other are found to be,

$$[A_1, A_2] = iA_3, [B_1, B_2] = iA_3, [A_1, B_1] = 0, [A_1, B_2] = iB_3, [A_1, B_3] = iB_2$$

If we define

$$J_l = \frac{1}{2}(A_l + B_l),\ K_l = \frac{1}{2}(A_l - B_l),\ l = 1, 2, 3$$

The commutators become

$$[J_1, J_2] = iJ_3, [K_1, K_2] = iK_3, [J_j, K_l] = 0,\ l, j = 1, 2, 3$$

This shows that each of the set J_1, J_2, J_3 and K_1, K_2, K_3 generates the group SO(3). So that

$$SO(4) = SO(3) \otimes SO(3)$$

A.7 Lie algebra

Let us consider a Lie group with r continuous parameters a_k having r generators $I_1, I_2, ..I_r$. We define the commutator of two generator I_k, I_l in the following way,

$$[I_k, I_l] = \sum_{j=1}^{r} c_{kl}^{j} I_j, \quad 1 \leqslant k, l \leqslant r; \tag{A.4}$$

The constant c_{kl}^{j} are called the structure constant. These are a characteristic property of the Lie group and don't depend on any particular representations of the generators. The commutator of pairs of generators of a Lie group determine the structure of the Lie group completely.

A Lie algebra is a real r-dimensional vector space L with elements $(x, y, z, ..)$ endowed with a law of composition for any two elements of L denoted by $[x, y]$ such that

- $[x, y] \in L$
- $[x, y] = -[y, x]$
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

for all $x, y, z \in L$. The law of composition $[x, y]$ is known as the commutator of x and y A set of r independent vectors of L is called a basis of the Lie algebra in analogy with the basis for a vector space. Since the commutator of the generators of a Lie group defined in (A.4) satisfies the above properties, we obtain the following relations among the structure constants:

$$c^j_{kl} = -c^j_{kl}$$
$$\sum_{m=1}^{r}[c^m_{kl}c^s_{jm} + c^m_{ij}c^s_{km} + c^m_{jk}c^s_{lm}] = 0 \tag{A.5}$$

Since the generators are Hermitian, (A.4) shows that the structure constants c^j_{kl} are purely imaginary. The importance of Lie algebra lies in the fact that we may generate a representation of a Lie group by considering a matrix representation of the Lie algebra. If we find a set of r square matrices of order p which satisfy (A.4) with the given structure constants, then, we can generate a p-dimensional representation of the Lie group. Therefore a representation of a Lie algebra can used to generate a representation of the associated Lie group[Jos88].

A.7.1 The special unitary group $SU(n)$

We now discuss the group $SU(n)$ [Bur85].The special unitary groups $SU(n)$ is encountered often in particle physics theories. It is a subgroup of the unitary group $U(n)$ consisting of all $n \times n$ unitary matrices, which is itself a subgroup of the general linear group $GL(n, \mathbb{C})$. It is $SU(2)$ in isospin invariance; $SU(3)$ in the eigthfold way; the standard gauge model of strong and electroweak interactions uses $SU(3) \times SU(3) \times U(1)$; the simplest grand unification group is $SU(5)$. $SU(n)$ is the group of $n \times n$ unitary matrices with unit determinant:$U^\dagger U = UU^\dagger$ and $detU = 1$. Any unitary matrix U can be written in terms of a hermitian matrix H as $U = e^{iH}$. From $det(e^A) = e^{tr(A)}$ and $detU = 1$, one has $trH = 0$. Since there are $n^2 - 1$ tracelessness hermitian matrices, an element of $SU(n)$ can be written as

$$U = \exp i \left\{ \sum_{a=1}^{n^2-1} \varepsilon_a J_a \right\} \tag{A.6}$$

where the ε_a are (real) group parameters. The J_a are group generators represented by traceless hermitian matrices. Only $n - 1$ of the $n^2 - 1$ generators are diagonal. We say that $SU(n)$ is a group of rank $n - 1$.

For instance the $SU(2)$ group has the generators known as the Pauli matrices given by:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.7}$$

These generators satisfy the commutation relations

$$[\sigma_j, \sigma_k] = 2i \sum_{i}^{3} \varepsilon_{kjl} \sigma_l$$

where ε_{jkl} is the fully antisymmetric tensor of rank 3 whose only nonvanishing elements are

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = -\varepsilon_{213} = -\varepsilon_{132} = -\varepsilon_{321} = 1$$

The components of the tensor ε_{jkl} multiplied by $2i$ are evidently the structure constants of $SU(2)$. The Lie algebra of $SU(2)$ is thus the set of all linear combinations of σ_x,σ_y and σ_z. The maximum number of mutually commuting generators of a Lie group is called is rank. The rank of $SO(3)$ is one since no two of generators L_x, L_y, L_z commute with each other. The rank of $SU(2)$ is also 1. An operator which commutes with all the generator of a Lie group is known as a Cassimir operator for the Lie group. The only one Casimir operator of $SO(3)$ is $\overrightarrow{L^2} = L_x^2 + L_y^2 + L_z^2$ and the Casimir operator of $SU(2)$ is similar $\overrightarrow{\sigma^2} = \sigma_x^2 + \sigma_y^2 + \sigma_z^2$. More reading see at [CF90],[PS91],[Jon91]

A.8 corresponding code for the last example in section 3.5

```
#In this python program, we consider just two rings, of radius r1=1.5, r2=1.
# we can generalise this program to the case of more than two rings.
from __future__ import division
from scipy import *
r=[1.5,1.]#this list contains the radius of rings.
x7=1.3
x5=1.5
def z(x7,r,x5):# here we define the function z.
sumz=0
for i in range (0,2):
num=x7**2-r[i]**2+x5**2
deno=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)**.5
sumz=sumz+.5*(-1)**(i+1)*(num/deno-1)
print sumz

def v(x7,r,x5):#here we define the function v.
sumv=0
for i in range (0,2):
num=x7**2+r[i]**2+x5**2
deno=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)**.5
sumv=sumv+.5*(-1)**(i)*(num/deno-1)
return sumv
def zr(x7,r,x5):#here we define the partial derivative of z with respect to r
tot1=0
```

```
for i in range (0,2):
#sum1=0
t1=x7*((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
t2=(x7**2-r[i]**2+x5**2)*(x7*(x7**2+r[i]**2+x5**2)-4*x7*r[i]**2)
t3=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
tot1=tot1+(-1)**(i+1)*(t1-t2)/t3**1.5
return tot1
def zy(x7,r,x5):#here we define the partial derivative of z with respect to y
tot2=0
for i in range(0,2):
u1=x5*((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
u2=x5*(x7**2-r[i]**2+x5**2)*(x7**2+r[i]**2+x5**2)
u3=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
tot2=tot2+(-1)**(i+1)*(u1-u2)/u3**1.5
return tot2
def vr(x7,r,x5):# partial derivative of v with respect to r
tot3=0
for i in range(0,2):
w1=x7*((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
w2=(x7**2+r[i]**2+x5**2)*(x7*(x7**2+r[i]**2+x5**2)-4*x7*r[i]**2)
w3=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
tot3=tot3+(-1)**(i)*(w1-w2)/w3**1.5
return tot3
def vy(x7,r,x5):# partial derivative of v with respect to y
tot4=0
for i in range(0,2):
z1=x5*((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
z2=x5*(x7**2+r[i]**2+x5**2)**2
z3=((x7**2+r[i]**2+x5**2)**2-4*(x7**2)*r[i]**2)
tot4=tot4+(-1)**(i)*(z1-z2)/z3**1.5
return tot4

#from this point we define the functions obtained on the page 28 of this work.
def h_2():
return -2*x5*sqrt(1-4**z(x7,r,x5)**2)
def gtty():
return (-2+8*z(x7,r,x5)**2+8*zy(x7,r,x5)*x5*z(x7,r,x5))/sqrt(1-4*z(x7,r,x5)**2)
def gttr():
return 8*x5*z(x7,r,x5)*zr(x7,r,x5)*(1-4*z(x7,r,x5)**2)**-.5
def gtphiy():
return -vy(x7,r,x5)*h_2()-v(x7,r,x5)*(-gtty())
def gtphir():
return -vr(x7,r,x5)*-h_2()-v(x7,r,x5)*(-gttr())
def gphiphiy():
return gtphiy()
```

```
def gphiphir():
return gtty()*v(x7,r,x5)**2-2*v(x7,r,x5)*vr(x7,r,x5)*(-h_2())+2*x7
def gyyy():
return (-1+4*z(x7,r,x5)**2+4*z(x7,r,x5)*x5*zy(x7,r,x5))/2*(x5**2)*
(1-4*z(x7,r,x5)**2)**1.5
def gyyr():
return 2*z(x7,r,x5)*zr(x7,r,x5)/x5*(1-4*z(x7,r,x5)**2)**1.5
def grrr():
return gyyr()
def grry():
return gyyy()
def g03y():
return zy(x7,r,x5)*(2/(1-2*z(x7,r,x5)**2))*((1+2*z(x7,r,x5))/
(1-2*z(x7,r,x5)))**-.5
def g03r():
return zr(x7,r,x5)*(-4*z(x7,r,x5)/(1-2*z(x7,r,x5)**2))*((1+2*z(x7,r,x5))/
(1-2*z(x7,r,x5)))**-.5
def g13y():
return zy(x7,r,x5)*(-4*z(x7,r,x5)/(1-2*z(x7,r,x5)))*((1-2*z(x7,r,x5))/
(1+2*z(x7,r,x5)))**-.5
def g13r():
return zr(x7,r,x5)*(-2/(1+2*z(x7,r,x5)**2))*((1-2*z(x7,r,x5))/
(1+2*z(x7,r,x5)))**-.5
#from this point we define the function called geodesics the main function.
#this function call others and use the initials value in w0, to find numerically
#the geodesics and to plot the graph.
def geodesic(w,t):
x1,x2,x3,x4,x5,x6,x7,x8,x9,x10,x11,x12=w
return array([x2,0,x4,0,x6,.5*gtty()*x2**2+.5*gphiphiy()*x4**2
+.5*gyyy()*x6**2+.5*grry()*x8**2+.5*g03y()*x10**2+.5*g13y()*x12**2
+gtphiy()*x2*x4,x8,.5*gttr()*x2**2+.5*gphiphir()*x4**2+.5*gyyr()*x6**2
+.5*grrr()*x8**2+.5*g03r()*x10**2+.5*g13r()*x12**2
+gtphir()*x2*x4,x10,0,x12,0])
w0=array([1.7,1.1,2.1,1.,1.3,1.,1.1,1.1,1.4,1.,1.2,1.9])
timepoints=arange(0.,6.28,.02)
trajectory=integrate.odeint(geodesic,w0,timepoints)
gplt.plot(trajectory[:,0:12:2])
gplt.xtitle('absciss')
gplt.ytitle('ordinate')
gplt.output("loises.eps", 'postscript')
```

Acknowledgements

First of all, I thank the almighty for the Grace and Mercy he has showered upon me. I heartily express my profound gratitude to Professor Neil Turok, the founder of AIMS, and to Professor Fritz Hahne, AIMS Director, for giving me this opportunity to become one of the students of this exciting institute. I am indebted to my supervisor, Professor Robert De Mello Koch, for developing my interest in this new field and for his tremendous support, help and friendly guidance. Special thanks to tutors for all the help they have given me. Special thanks to all of the staff, both academic and non-academic, for creating and enabling an environment conducive for study. Special thanks for my entire family, my wife Meda and my kid for their sacrifices and support. All my best wishes to my dear friends.

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