Partial Metrics

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Abstract

In this essay, the concept of partial metric spaces, or equivalently, weightable quasi-metrics, are investigated to generalise metric spaces. In particular, the self-distance for any point need not be equal to zero. The main part of this work concentrates on the theory of partial metric spaces by referring to their induced metric spaces. Working in a complete space motivates the completion of partial metric spaces, followed by the extension of Banach's fixed point theorem to these spaces. Correspondence between partial metrics and semivaluations is taken into account since it is a powerful tool in computer science for the study of semantics and domains.

Keywords: Partial metric, metric, topology, completion, fixed point, semivaluation.

Voarakitra ato anatin' ity asa ity ny fomba fahitana ny *ampahan-drefy*, na *saika-refy* azo lanjaina izay mitovy aminy, mba hijerena amin' ny ankapobeny ny *refy*. Singanina manokana etoana fa ny elanelana misy amina teboka iray miala eo aminy ihany dia tsy voatery ho aotra. Ny votoatin' ity asa ity dia miompana amin' ny fomba fijery ny saha misy ny *ampahan-drefy* raha mitaha amin' ny saha misy *refy* aterany ihany. Ny fiasana amina saha feno iray no antony hamenoina ny saha misy ny *ampahan-drefy*, manaraka ny fanitarana ny lalànan' i *Banach* momba ny teboka tsy mihetsika ao anatin' ilay saha fenon' ny *ampahan-drefy*. Omena lanja manokana ny fifandraisana misy amina *ampahan-drefy* sy ny *semivaloasona* izay fitaovana iray mahomby sy manan-danja amin' ny ny siansa momba ny solo-saina ho fianarana ny semantika sy domenina.

Teny mpamaritra: ampahan-drefy, refy, topolojia, famenoina, teboka tsy mihetsika, semivaloasona.

Contents

Introduction

Partial metric spaces were originally developed by Matthews [Mat94, Mat] to provide mechanism generalising metric space theories. This is a relatively new field and has vast application potentials [KMP05] in the study of computer domains and semantics [O'N95]. There have been different approaches in this area [KPS06, RS03] when it comes to applying the developing mathematical concepts to computer science [Was02, Sch03].

This report is entirely mathematical, and requires a background in metric spaces and general topology [Heu75, Hu66, Man62]. We will bring together the necessary basic concepts to generalise the metric spaces and their topological properties into partial metric spaces, under the bewildering axiom that the self distance of any point need not be zero.

This essay focuses on the basic theory of partial metric spaces, especially on their completion, and also on Banach's fixed point theorem which is extended to their spaces. Correspondence between semivaluations and partial metrics is also considered, and some examples with applications from [O'N95] are carefully studied.

Overall, we only study the mathematical approach of partial metric concepts. In Chapter 1, we start with the topological properties of partial metric spaces and contrast metric spaces. We then examine the correspondence between partial metrics and weightable quasi-metrics [Mat94].

In Chapter 2, we proceed to characterise partial metric spaces through the properties their induced metric spaces have [Mat94]. We then turn to construct their completion [O'N95] which is similar to the metric spaces case, and finally we extend Banach's fixed point theorem to complete partial metric spaces [Val05].

In Chapter 3, we prove the correspondence between co-weightable quasi-metrics and partial metrics. We also introduce the notion of the semilattices and come to study the correspondence between partial metric semilattices and semivaluations [Sch04, RS05].

In Chapter 4, we consider the examples of the powerset of natural numbers and the Baire partial metric space [O'N95] which have applications in computer science.

Though partial metrics are used in computer science to study semantics and domain theory [O'N95, But02], the cope of this report will oblige us to concentrate on overviews of general mathematical properties of partial metric spaces. In particular, we will focus on the way in which they generalise metric space notions.

1. Partial metric spaces

In spite of its relevant application in computer science [Mat94, Eda97], this Chapter only focuses on the mathematical approach for partial metrics [Mat94, O'N95, KPS06] described as the generalisation of metrics.

1.1 Basic concepts

Definition 1.1.1. [Mat94] A *partial metric* on a non-empty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that, for any $x, y, z \in X$,

P1. $p(x, x) \leq p(x, y)$.

P2. $x = y$ if, and only if, $p(x, x) = p(x, y) = p(y, y)$.

P3. $p(x, y) = p(y, x)$.

P4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Note that the self distance of any point need not be zero, hence the idea of generalising metrics so that a *metric* on a non-empty set X is precisely a partial metric p on X such that for any $x \in X$, $p(x, x) = 0$.

Similar to the case of metric space, a *partial metric space* is a pair (X, p) consisting of a non-empty set X and a partial metric p on X.

Definition 1.1.2. Let (X, p) be a partial metric space. For any $x \in X$ and $\varepsilon > 0$, we define respectively the *open* and *closed ball* for the partial metric p by setting

$$B_\varepsilon(x) = \{y \in X : p(x, y) < \varepsilon\}, \quad \overline{B}_\varepsilon(x) = \{y \in X : p(x, y) \leq \varepsilon\}$$

Remark 1.1. Contrary to the metric space case, some open balls may be empty. As an example, in a partial metric space (X, p), the open balls $B_{p(x,x)}(x)$ are empty for any $x \in X$.

Example 1.1. [O'N95] Consider the function $p : \mathbb{R}^- \times \mathbb{R}^- \longrightarrow \mathbb{R}^+$ defined by $p(x, y) = -\min\{x, y\}$ for any $x, y \in X$. The pair $(\mathbb{R}^-, p)$ is a partial metric space for which p is called the *usual partial metric* on $\mathbb{R}^-$, and where the *self-distance* for any point $x \in \mathbb{R}^-$ is its absolute value.

Indeed, for any $x, y, z \in \mathbb{R}^-$,
P1. $\min\{x, y\} \leq x$ so $p(x, y) \geq p(x, x) = -x$,
P2. suppose that $p(x, x) = p(x, y) = p(y, y)$, it then follows that $-x = -y$, hence $x = y$,
P3. it is obvious that $p(x, y) = p(y, x)$,
P4. one verifies that $\min\{x, z\} \geq \min\{x, y\} + \min\{y, z\} - \min\{y, y\}$ by considering the cases $y \leq x \leq z, \;\; x \leq y \leq z$ and $x \leq z \leq y$, hence $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The open balls are of the form $B_\varepsilon(x) = \{y \in \mathbb{R}^- : -\min\{x, y\} < \varepsilon\} = (-\varepsilon, 0)$ with $x \geq -\varepsilon$ otherwise, if $x < -\varepsilon$ then $p(x, x) = -x > \varepsilon$ and $B_\varepsilon(x) = \emptyset$. Suppose that $y \in B_\varepsilon(x)$, then $-\min\{x, y\} < \varepsilon$ which implies that $\min\{x, y\} > -\varepsilon$, hence $y > -\varepsilon$.

Example 1.2. [O'N95] Similar to the previous example, let $p' : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be defined by $p'(x, y) = \max\{x, y\}$ for any $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p')$ is a partial metric space where the self-distance for any point $x \in \mathbb{R}^+$ is its value itself.

One verifies that for any $x, y, z \in \mathbb{R}^+$,
P1. $\max\{x, y\} \geq y$ so $p'(x, y) \geq p'(x, x)$,
P2. suppose that $p'(x, x) = p'(x, y) = p'(y, y)$, it then follows that $x = y$,
P3. it is obvious that $p'(x, y) = p'(y, x)$,
P4. one verifies that $\max\{x, z\} \leq \max\{x, y\} + \max\{y, z\} - \max\{y, y\}$ by considering the cases $y \leq x \leq z, \quad x \leq y \leq z$ and $x \leq z \leq y$, hence $p'(x, z) \leq p'(x, y) + p'(y, z) - p'(y, y)$.

The open balls are of the form $B_\varepsilon(x) = \{y \in \mathbb{R}^+ : \max\{x, y\} < \varepsilon\} = (0, \varepsilon)$ with $x \leq \varepsilon$ otherwise, if $x \geq \varepsilon$ then $B_\varepsilon(x) = \emptyset$. Suppose that $y \in B_\varepsilon(x)$, then $\max\{x, y\} < \varepsilon$ which implies that $y < \varepsilon$.

Theorem 1.1.1. *In a partial metric space (X, p), the set of open balls is the basis of a T_0 topology on X, called the partial metric topology and denoted by $\mathcal{T}[p]$.*

Proof. Since $B_{p(x,x)+1}$ contains x, one easily has that $X = \bigcup_{x \in X} B_{p(x,x)+1}(x)$. Let $B_\varepsilon(x)$, $B_\delta(y)$ be open balls in (X, p) and $z \in B_\varepsilon(x) \cap B_\delta(y)$. Consider $\eta = p(z, z) + \min\{\varepsilon - p(x, z), \delta - p(y, z)\}$, then $z \in B_\eta(z)$ since one has $\varepsilon - p(x, z) > 0, \quad \delta - p(y, z) > 0$, thus $p(z, z) < \eta$.

Now, suppose that $z' \in B_\eta(z)$, we prove that $z' \in B_\varepsilon(x) \cap B_\delta(y)$. From P4., one has that

$$\begin{array}{rcl} p(x, z') \leq & p(x, z) + p(z, z') - p(z, z) \\ < & \eta - p(z, z) + p(x, z) \\ < & \varepsilon - p(x, z) + p(x, z) = \varepsilon. \end{array}$$

Hence $z' \in B_\varepsilon(x)$. Similarly, we prove that $z' \in B_\delta(y)$ since

$$\begin{array}{rcl} p(y, z') \leq & p(y, z) + p(z, z') - p(z, z) \\ < & \eta - p(z, z) + p(y, z) \\ < & \delta - p(y, z) + p(y, z) = \delta. \end{array}$$

Consequently, $z' \in B_\varepsilon(x) \cap B_\delta(y)$, hence $B_\eta(z) \subset B_\varepsilon(x) \cap B_\delta(y)$. The set of open balls is then a basis for a topology denoted by $\mathcal{T}[p]$ on X.

To see that the topology is T_0, let us suppose that $x, y \in X$ are distinct points and that $p(x, x) < p(x, y)$. Considering $\varepsilon = (p(x, y) + p(x, x))/2 > 0$, one has $x \in B_\varepsilon(x)$ but $y \notin B_\varepsilon(x)$ since one has $B_\varepsilon(x) = \{z \in X : p(x, z) < \varepsilon\}$, then $2p(x, x) < p(x, y) + p(x, x)$ but $2p(x, y) \not< p(x, y) + p(x, x)$. ■

The study of partial metrics lead us to the interesting concept of partial orders which often arise in computer science for the study of domains [Was02, But02].

Definition 1.1.3. A *partial order* is a pair $(X, \leq)$ consisting of a non-empty set X and a relation $\leq$ on X such that, for any $x, y, z \in X$

1. $x \leq x$ (reflexivity),
2. if $x \leq y$ and $y \leq x$ then $x = y$ (antisymmetry),
3. if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity).

Remark 1.2. [O'N95] Once we have a T_0 topology on a non-empty set X, there exists a natural partial order $\sqsubseteq$ called the specialisation order on X, defined for any $x, y \in X$ by $x \sqsubseteq y \Longleftrightarrow x \in U$, U open, imply $y \in U$ (x in the closure of $\{y\}$).

Proposition 1.1.2. *[O'N95] In a partial metric space (X, p), $x \sqsubseteq_P y$ if, and only if, $p(x,x) = p(x,y)$. In $(\mathbb{R}^-, p)$, $x \sqsubseteq_p y$ if, and only if, $x \leq y$. In $(\mathbb{R}^+, p')$, $x \sqsubseteq_p y$ if, and only if, $x \geq y$.*

Proof. Suppose first that $x \sqsubseteq_p y$, then for all $\varepsilon > 0, y \in B_\varepsilon(x)$ and one has $p(x,x) \leq p(x,y) < \varepsilon < p(x,x) + \varepsilon$ thus $p(x,y) = p(x,x)$.

Conversely, suppose that $p(x,x) = p(x,y)$ and $x \in U$ with $U \in \mathcal{T}[p]$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. However, $p(x,y) = p(x,x) < \varepsilon$ implies $y \in B_\varepsilon(x)$ so $y \in U$. Since this holds for any $U \in \mathcal{T}[p]$, we see that $x \sqsubseteq_p y$.

In $(\mathbb{R}^-, p)$, we have that $p(x,x) = p(x,y) \Longleftrightarrow -x = -\min\{x,y\} \Longleftrightarrow x \leq y$; in $(\mathbb{R}^+, p')$, $p'(x,x) = p'(x,y) \Longleftrightarrow x = \max\{x,y\} \Longleftrightarrow x \geq y$. ■

Example 1.3. [Mat94] The interval domain.
Let us consider the set $I = \{[a,b] : a \leq b, \quad a, b \in \mathbb{R}\}$ of closed intervals in $\mathbb{R}$ and define $p : I \times I \longrightarrow \mathbb{R}^+$ by setting $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (I, p) is a partial metric space.

Indeed, for any $a, b, c, d, e, f \in \mathbb{R}$,
P1. one verifies that $\max\{b,d\} - \min\{a,c\} \geq b - a$, hence $p([a,b],[c,d]) \geq p([a,b],[a,b])$,
P2. suppose that $p([a,b],[a,b]) = p([a,b],[c,d]) = p([c,d],[c,d])$.
Then $b - a = d - c = \max\{b,d\} - \min\{a,c\}$. So $[a,b]$ and $[c,d]$ have the same length. Suppose that $\max\{b,d\} = b$, then $\min\{a,c\} = c$ thus $[a,b] \subset [c,d]$. Since they have the same length, they must be equal, that is $[a,b] = [c,d]$.
P3. It is clear that $p([a,b],[c,d]) = ([a,b],[c,d])$.
P4. Consider $p([a,b],[e,f]) + p([e,f],[c,d]) - p([e,f],[e,f]) = \max\{b,f\} - \min\{a,e\} + \max\{f,d\} - \min\{c,e\} - f + e$. One verifies that $\max\{b,f\} + \max\{f,d\} - f \geq \max\{b,d\}$ and $-\min\{a,e\} - \min\{e,c\} - e \geq -\min\{a,c\}$, hence we have that

$p([a,b],[c,d]) \leq p([a,b],[e,f]) + p([e,f],[c,d]) - p([e,f],[e,f])$.

Given $\varepsilon > 0$ and $[a,b] \in I$, considering an element $[c,d] \in I$, one has that

$$\begin{aligned} [c,d] \in B_\varepsilon([a,b]) &\Longleftrightarrow \quad p([a,b],[c,d]) < \varepsilon \\ &\Longleftrightarrow \quad \max\{b,d\} - \min\{a,c\} < \varepsilon. \end{aligned}$$

In particular, $[a,b] \in B_\varepsilon([a,b])$ if, and only if, $b-a<\varepsilon$. The self-distance $p([a,b],[a,b])$ for any $a,b\in\mathbb{R}, a\le b$ is the length $b-a$ of the interval $[a,b]$.

Here, $[a,b] \sqsubseteq [c,d]$ if, and only if, $[c,d]\subseteq[a,b]$. Indeed, $p([a,b],[a,b]) = p([a,b],[c,d])$ implies $b-a = \max\{b,d\} - \min\{a,c\}$. Suppose that $d>b$, then $b-a = d-\min\{a,c\}$ and $\min\{a,c\}-a = d-b>0$, hence $\min\{a,c\}>a$ which is impossible, then $d\le b$. Similarly, one proves that $a\le c$, otherwise $b-a = \max\{b,d\}-c$ implies $b>\max\{b,d\}$ which is impossible. Consequently, $[c,d]\subseteq[a,b]$ if, and only if, $p([a,b],[a,b]) = p([a,b],[c,d])$.

Definition 1.1.4. A sequence (x_n) in a partial metric space (X,p) converges to $x\in X$, and one writes $\lim\limits_{n\to+\infty} x_n = x$ if, for any $\varepsilon>0$ such that $x\in B_\varepsilon(x)$, there exists $N\ge 1$ so that for any $n\ge N, x_n\in B_\varepsilon(x)$.

Proposition 1.1.3. *Suppose that (x_n) is a sequence in a partial metric space (X,p) and $x\in X$. Then $x_n\to x$ if, and only if, $\lim\limits_{n\to+\infty} p(x_n,x) = p(x,x)$.*

Proof. Suppose first that $x_n\to x$, then for any $\varepsilon>0$, there exists $N\ge1$ such that for any $n\ge N, x_n\in B_\varepsilon(x)$ that is $0\le p(x_n,x)<\varepsilon$. Since $p(x,x)\le p(x_n,x)<\varepsilon<p(x,x)+\varepsilon$, we have that $0\le p(x_n,x)-p(x,x)\le\varepsilon$. When $n\to+\infty$ one obtains $0\le \lim\limits_{n\to+\infty} p(x_n,x)-p(x,x)<\varepsilon$, which holds for every $\varepsilon>0$, hence $\lim\limits_{n\to+\infty} p(x_n,x) = p(x,x)$.

Conversely, if $\lim\limits_{n\to+\infty} p(x_n,x) = p(x,x)$, then for any $\varepsilon_1>0$, there exists $N\ge1$ such that for any $n\ge N, p(x_n,x)<p(x,x)+\varepsilon_1$. Let $\varepsilon>0$, one takes $\varepsilon_1 = \varepsilon - p(x,x)$. Then $x_n\in B_\varepsilon(x)$ so that for any $\varepsilon>0$, there exists $N\ge1$ such that for any $n\ge N$, $x_n\in B_\varepsilon(x)$, hence $x_n\to x$. ∎

Definition 1.1.5. Suppose that (x_n) is a sequence in a partial metric space (X,p), then we define $L(x_n)$ to be the set of limit points of (x_n).

As an example, in the usual partial metric $(\mathbb{R}^-,p)$, the sequence $(-\frac{1}{n})$ has $L(-\frac{1}{n}) = (-\infty,0)$. Indeed, suppose that $x<0$, then $p(-\frac{1}{n},x) = -\min\{-\frac{1}{n},x\}$. Let be $\varepsilon>0$ such that $x\in B_\varepsilon(x)$, then there exists $N\ge1$ such that $-\frac{1}{n}>x$, hence $\min\{-\frac{1}{n},x\} = x$ and $p(-\frac{1}{n},x) = -x = p(x,x)<\varepsilon$. Consequently $-\frac{1}{n}\in B_\varepsilon(x)$, hence $(-\frac{1}{n})$ converges to x.

Proposition 1.1.4. *Let (x_n) be a sequence in a partial metric space (X,p). If a point $a\in L(x_n)$ and $a'\sqsubseteq a$, then $a'\in L(x_n)$.*

Proof. Recall that $x_n\to a'$ if, and only if, $\lim\limits_{n\to+\infty} p(x_n,a') = p(a',a')$ and $a'\sqsubseteq a$ if, and only if $p(a',a') = p(a',a)$. Suppose that $p(a',a') = p(a',a)$, since $p(x_n,a')\le p(x_n,a)+p(a,a')-p(a,a)$ for any $n\in\mathbb{N}$, when n tends to $+\infty$, the terms in the right tend to $p(a',a')$ which imply that $\lim\limits_{n\to+\infty} p(x_n,a')\le p(a',a')$. Since for any $n\in\mathbb{N}$, $p(x_n,a')\ge p(a',a')$, we have that $\lim\limits_{n\to+\infty} p(x_n,a')\ge p(a',a')$, and finally $\lim\limits_{n\to+\infty} p(x_n,a') = p(a',a')$. ∎

1.2 Partial metrics and weightable quasi-metrics

Recall that a *quasi-metric* on a non-empty set X is a function $q : X \times X \longrightarrow \mathbb{R}^+$ satisfying

Q1. For any $x, y \in X$, $x = y \iff q(x, y) = 0$ and $q(y, x) = 0$.

Q2. For any $x, y, z \in X$, $q(x, y) \leq q(x, z) + q(z, y)$.

A *quasi-metric space* is a pair (X, q) where q is a quasi-metric on the non-empty set X. For any $a \in X$ and $r > 0$, an *open ball* on a (X, q) is of the form $B_r^q(a) = \{x \in X : q(a, x) < r\}$ and the set of all open balls on X forms a topological basis B by setting

$$\mathcal{B}^q = \{B_r^q(a) : a \in X \text{ and } r \in \mathbb{R}^+\}$$

by which is generated a topology $\mathcal{T}[\mathcal{B}^q]$ on X.

The induced natural order on X generated by a quasi-metric q is a relation $\sqsubseteq_q$ such that for any $x, y \in X$, $x \sqsubseteq_q y$ if, and only if $q(x, y) = 0$.

Definition 1.2.1. A *weighted quasi-metric* on a non-empty set X is a pair (q, ω) consisting of a quasi-metric q on X and a *weighting* function ω, such that

$\omega : X \longrightarrow \mathbb{R}^+$ and $q(x, y) + \omega(x) = q(y, x) + \omega(y)$ for any $x, y \in X$.

A *weighted quasi-metric space* is a triple (X, q, ω) where (X, q) is a quasi-metric space and (q, ω) is a weighted quasi-metric.

We say that a quasi-metric q is *weightable* if there exists a function $\omega : X \longrightarrow \mathbb{R}^+$ such that (q, ω) is a weighted quasi-metric. The quasi-metric space (X, q) is then said to be *weightable*.

The elementary example is a metric space which can be viewed as a weighted quasi-metric space, where the weighting function is constant (usually zero). The following theorems give the correspondence between partial metrics and weighted quasi-metrics.

Theorem 1.2.1. *[Mat94] For each partial metric $p : X \times X \longrightarrow \mathbb{R}^+$ on a non-empty set X, the function $q : X \times X \longrightarrow \mathbb{R}^+$ defined by $q(x, y) ::= p(x, y) - p(x, x)$ is a weighted quasi-metric with weighting function $\omega : X \longrightarrow \mathbb{R}^+$, such that for any $x \in X, \omega(x) = p(x, x)$. Moreover, $\mathcal{T}[q] = \mathcal{T}[p]$ and for any $x, y \in X$, $x \sqsubseteq_q y$ if, and only if, $x \sqsubseteq_p y$.*

Proof. Let us show first that q is a quasi-metric. It is clear that for any $x, y \in X, q(x, x) = 0$ and $q(x, y) \geq 0$. Suppose that $q(x, y) = q(y, x) = 0$, then $p(x, y) - p(x, x) = p(y, x) - p(y, y) = 0$ hence $p(x, y) = p(x, x) = p(y, y) = 0$ which implies that $x = y$. Now, since for any $x, y, z \in X$, one has $p(x, z) - p(x, x) \leq p(x, y) - p(x, x) + p(y, z) - p(y, y)$, thus $q(x, z) \leq q(x, y) + q(y, z)$. Hence q is a quasi-metric on X.

For any $x, y \in X$, since $q(x, y) = p(x, y) - \omega(x)$ and $q(y, x) = p(y, x) - \omega(y)$, taking into account that $p(x, y) = p(y, x)$, it follows that $q(x, y) + \omega(x) = q(y, x) + \omega(y)$, thus (q, ω) is a weighted quasi-metric on X.

Now, let us prove that $\mathcal{T}[q] = \mathcal{T}[p]$. For any $x \in X$ and $\varepsilon > 0$ such that $x \in B^p_\varepsilon(x)$, if $y \in B^p_\varepsilon(x)$, then $q(x,y) = p(x,y) - p(x,x) < \varepsilon - p(x,x)$ so that $y \in B^q_{\varepsilon - p(x,x)}(x)$. Conversely, if $y \in B^q_{\varepsilon - p(x,x)}(x)$, then $y \in B^p_\varepsilon(x)$, hence for any $x \in X$, and $\varepsilon > p(x,x)$, $B^p_\varepsilon(x) = B^q_{\varepsilon - p(x,x)}(x)$.

For any $x \in X$ and for any $0 < \varepsilon \leq p(x,x)$, one has $B^p_\varepsilon(x) = \emptyset$.

For any $x \in X$ and $\varepsilon > 0$, if $y \in X, y \in B^q_\varepsilon(x)$, then $p(x,y) - p(x,x) < \varepsilon$ which implies that $y \in B^p_{\varepsilon + p(x,x)}(x)$. Conversely, if $y \in B^p_{\varepsilon + p(x,x)}(x,x)$, then it is clear that $q(x,y) < \varepsilon$ so that $y \in B^q_\varepsilon(x)$. Hence for any $x \in X$ and $\varepsilon > 0$, $B^q_\varepsilon(x) = B^p_{\varepsilon + p(x,x)}(x,x)$.

Finally, for any $x, y \in X$, $p(x,x) = p(x,y)$ if, and only if, $q(x,y) = 0$. Hence $x \sqsubseteq_p y$ if and only if, $x \sqsubseteq_q y$. ■

Theorem 1.2.2. *[Mat94] For each weighted quasi-metric (q, ω) on a non-empty set X, the function $p : X \times X \longrightarrow \mathbb{R}^+$ defined by $p(x,y) ::= q(x,y) + \omega(x)$ for any $x, y \in X$ is a partial metric on X such that for any $x \in X, p(x,x) = \omega(x)$. Moreover, $\mathcal{T}[p] = \mathcal{T}[q]$ and for any $x, y \in X$, $x \sqsubseteq_p y$ if, and only if, $x \sqsubseteq_q y$.*

Proof. It is clear that for any $x \in X$, one has $p(x,x) = \omega(x)$.
P1. For any $x, y \in X, 0 \leq q(x,y)$, then $\omega(x) \leq q(x,y) + \omega(x)$. Since $p(x,x) = \omega(x)$ for any $x \in X$, one has $p(x,x) \leq p(x,y)$.
P2. If for any $x, y \in X$, $p(x,x) = p(x,y) = p(y,y)$, then $\omega(x) = q(x,y) + \omega(x) = \omega(y)$. Since (q, ω) is a weighted quasi-metric, one has $\omega(x) = q(y,x) + \omega(y) = \omega(y)$. Thus $q(x,y) = q(y,x) = 0$ and hence $x = y$. The reverse is obvious.
P3. Since (q, ω) is a weighted quasi-metric, then for any $x, y \in X$, $q(x,y) + \omega(x) = q(y,x) + \omega(y)$ which implies that $p(x,y) = p(y,x)$.
P4. For any $x, y, z \in X$ we have that

$$\begin{aligned} q(x,z) &\leq q(x,y) + q(y,z), \\ q(x,z) + \omega(x) &\leq q(x,y) + \omega(x) + q(y,z) + \omega(y) - \omega(y), \text{and} \\ p(x,z) &\leq p(x,z) + p(y,z) - p(y,y). \end{aligned}$$

Thus p is a partial metric on X.

Let us prove that $\mathcal{T}[p] = \mathcal{T}[q]$. For any $x \in X$ and $\varepsilon > 0$ such that $x \in B^p_\varepsilon(x)$, if $y \in X$, and $y \in B^p_\varepsilon(x)$, that is $p(x,y) < \varepsilon$, then $q(x,y) < \varepsilon - \omega(x)$ and hence $y \in B^q_{\varepsilon - \omega(x)}(x)$. Conversely, if $y \in B^q_{\varepsilon - \omega(x)}(x)$, then $p(x,y) < \varepsilon$ so that $y \in B^p_\varepsilon(x)$. Hence $B^p_\varepsilon(x) = B^q_{\varepsilon - \omega(x)}(x)$.

For any $x \in X$ and $0 < \varepsilon \leq \omega(x)$, $B^p_\varepsilon(x) = \emptyset$.

For any $x \in X$ and $\varepsilon > 0$, if $y \in X, y \in B^q_\varepsilon(x)$, that is $q(x,y) < \varepsilon$, then $p(x,y) < \varepsilon + \omega(x)$ hence $y \in B^p_{\varepsilon + \omega(x)}(x)$. Conversely, if $y \in B^p_{\varepsilon + \omega(x)}(x)$, then $q(x,y) < \varepsilon$ so that $y \in B^q_\varepsilon(x)$. Hence $B^q_\varepsilon(x) = B^p_{\varepsilon + \omega(x)}(x)$.

Finally, for any $x, y \in X$, one has $p(x,x) = p(x,y)$ if, and only if, $q(x,y) = 0$. Then $x \sqsubseteq_p y$ if, and only if, $x \sqsubseteq_q y$. ■

Remark 1.3. The two last theorems demonstrated that there is an algebraic equivalence between the partial metrics class and the class of weighted quasi-metrics, in the sense that one can define a weighted quasi-metric from a partial metric; conversely, given a weighted quasi-metric, one can obtain a partial metric.

Remark 1.4. [Mat94] Not every quasi-metric is weightable. To see this, consider the function $q : \{a, b, c\} \times \{a, b, c\} \longrightarrow \{0, 1, 2, 3\}$ such that $q(a, b) = 0,\ \ q(b, a) = 2,\ \ q(a, c) = 1,\ \ q(c, a) = 1,\ \ q(b, c) = 3,\ \ q(c, b) = 0$ with $q(x, x) = 0$ for any $x \in \{a, b, c\}$. Then q is a quasi-metric on $\{a, b, c\}$ which is not weightable.

Proof.
Q1. Suppose that $x, y \in \{a, b, c\}$ but $x \neq y$, then $q(x, y) \neq q(y, x)$ or $q(x, y) \neq 0$ or $q(y, x) \neq 0$.

Q2. By construction, the triangle inequality holds:

$$\begin{aligned} q(a,b) \leq\ & q(a,c) + q(c,b) \text{ and } & q(b,a) \leq q(b,c) + q(c,b), \\ q(a,c) \leq\ & q(a,b) + q(b,c) \text{ and } & q(c,a) \leq q(c,b) + q(b,a), \\ q(b,c) \leq\ & q(b,a) + q(a,c) \text{ and } & q(c,b) \leq q(c,a) + q(a,b). \end{aligned}$$

Suppose by way of contradiction, that there exists a weighting function $\omega : \{a, b, c\} \longrightarrow \{0, 1, 2, 3\}$ for q. Then one has the following equalities:

$$\begin{aligned} \omega(b) + q(b,c) =& (\omega(b) + q(b,a)) + 1 \\ =& \omega(a) + q(a,b) + 1 \text{ (since } (q,\omega) \text{ is a weighted quasi-metric)} \\ =& \omega(a) + q(a,c) \\ =& \omega(c) + q(c,a) \text{ (since } (q,\omega) \text{ is a weighted quasi-metric)} \\ =& (\omega(c) + q(c,b)) + 1 \\ =& \omega(b) + q(b,c) + 1 \text{ (since } (q,\omega) \text{ is a weighted quasi-metric), contradiction.} \end{aligned}$$

■

2. Characterisations

In this chapter, we consider the relation between partial metrics and metrics [Mat94, O'N95] in order to characterise a partial metric space through its induced metric space.

2.1 Metric and partial metric spaces

Definition 2.1.1. A *weighted metric* on a non-empty set X is a pair(d, ω) consisting of a metric $d : X \times X \longrightarrow \mathbb{R}^+$ and a *weighting* function $\omega : X \longrightarrow \mathbb{R}^+$ such that for any $x, y \in X$, $d(x, y) \geq \omega(x) - \omega(y)$. A *weighted metric space* is a triple (X, d, ω) such that (X, d) is a metric space and (d, ω) a weighted metric on X.

A metric d on a non-empty set X is *weightable* if there exists a weighting function ω such that (d, ω) is a weighted metric. Then the pair (X, d) is said to be weightable.

The following theorems relate partial metrics to their induced weighted metrics [Mat94, O'N95] by which we refer when studying partial metric spaces.

Theorem 2.1.1. *[Mat94] Given a partial metric space (X, p), the pair (p^m, ω) of functions specified by $p^m : X \times X \longrightarrow \mathbb{R}^+$ and $\omega : X \longrightarrow \mathbb{R}^+$ such that for any $x, y \in X$, $p^m(x, y) ::= 2p(x, y) - p(x, x) - p(y, y)$, and for any $x \in X, \omega(x) ::= p(x, x)$, is a weighted metric. Moreover, $\mathcal{T}[p] \subseteq \mathcal{T}[p^m]$ such that for any $x, y \in X$, one has $p(x, y) = (p^m(x, y) + \omega(x) + \omega(y))/2$; p^m is then called the induced metric of p.*

Proof. In order to prove Theorem 2.1.1, we need Theorem 1.2.1 in chapter 1, stating that the function $q : X \times X \longrightarrow \mathbb{R}^+$ defined by $q(x, y) ::= p(x, y) - p(x, x)$ for any $x, y \in X$, is a quasi-metric such that $\mathcal{T}[q] = \mathcal{T}[p]$. Thus for any $x, y \in X$, $p^m(x, y) = q(x, y) + q(y, x)$, and it follows that p^m is a metric on X.

For any $x, y \in X$, $p^m(x, x) = q(x, x) - q(x, x) = 0$, and if $p^m(x, y) = 0 = q(x, y) + q(y, x)$, then $q(x, y) = q(y, x) = 0$ which implies that $x = y$.

It is clear that for any $p^m(x, y) = p^m(y, x) \geq 0$ for any $x, y \in X$.

Since q verifies the triangle inequality, so does p^m.

Now, let us prove that $\mathcal{T}[q] = \mathcal{T}[p] \subseteq \mathcal{T}[p^m]$. Since for any $x, y \in X, q(x, y) \leq p^m(x, y)$, then for any $\varepsilon > 0$ and $x \in X$, we have that $B_\varepsilon^{p^m}(x) \subseteq B_\varepsilon^q(x)$, hence $B_\varepsilon^q(x) \in \mathcal{T}[p^m]$. Consequently, one has $\mathcal{T}[q] = \mathcal{T}[p] \subseteq \mathcal{T}[p^m]$.

Finally, for any $x, y \in X$, $p(x, x) \leq p(x, y)$, then $2p(x, x) \leq 2p(x, y)$. Hence one obtains that $2p(x, x) - p(x, x) - p(y, y) \leq 2p(x, y) - p(x, x) - p(y, y)$ and then $\omega(x) - \omega(y) \leq p^m(x)$. ■

For example, in $(\mathbb{R}^-, p)$ where p is the usual partial metric on $\mathbb{R}^-$, we obtain the usual distance in $\mathbb{R}^-$ since for any $x, y \in \mathbb{R}^-$, $p^m(x,y) = 2p(x,y) - p(x,x) - p(y,y) = x + y - 2\min\{x,y\} = |x-y|$.

Theorem 2.1.2. *[Mat94] Given a weighted metric (d,ω) on a non-empty set X, the function $p : X \times X \longrightarrow \mathbb{R}^+$, where for any $x, y \in X$, $p(x,y) ::= (d(x,y) + \omega(x) + \omega(y))/2$, is a partial metric on X such that $d = p^m$ and for any $x \in X$, $\omega(x) = p(x,x)$.*

Proof. P1. For any $x, y \in X$, $p(x,y) = (d(x,y) + \omega(x) + \omega(y))/2 \geq \omega(x) = p(x,x)$ since $d(x,y) \geq \omega(x) - \omega(y)$.

P2. Suppose that $p(x,x) = p(x,y) = p(y,y)$, one then has that $\omega(x) = \omega(x) + d(x,y)$, thus $d(x,y) = 0$ and $x = y$. The converse is obvious.
P3. It is clear that for any $x, y \in X$, $p(x,y) = p(y,x)$.
P4. For any $x, y, z \in X$, we have that

$$\begin{aligned} p(x,z) =& (d(x,z) + \omega(x) + \omega(z))/2 \\ \leq& (d(x,y) + d(y,z) + \omega(x) + \omega(z))/2 \\ \leq& (d(x,y) + \omega(x) + \omega(y))/2 + (d(y,z) + \omega(y) + \omega(z))/2 - \omega(y) \\ \leq& p(x,y) + p(y,z) - p(y,y). \end{aligned}$$

■

Definition 2.1.2. [O'N95] Suppose that (X_1, p_1) and (X_2, p_2) are partial metric spaces with induced metrics p_1^m and p_2^m respectively. Then the function $f : (X_1, p_1) \longrightarrow (X_2, p_2)$ is said to be *continuous* if both $f : (X_1, \mathcal{T}[p_1]) \longrightarrow (X_2, \mathcal{T}[p_2])$ and $f : (X_1, p_1^m) \longrightarrow (X_2, p_2^m)$ are respectively continuous in the sense of topological spaces and metric spaces continuity.

Definition 2.1.3. If (x_n) is a sequence in a partial metric space (X,p), then x is a *proper limit* of (x_n), written $x_n \rightarrow x$ (properly) if $x_n \rightarrow x$ in (X, p^m). If a sequences has a proper limit, then one says that the sequence is *properly convergent*.

Lemma 2.1.3. *[O'N95] Suppose that (x_n) is a sequence in a partial metric space (X,p) and $x \in X$. Then $x_n \rightarrow x$ (properly) if, and only if, $\lim\limits_{n\to+\infty} p(x_n, x) = \lim\limits_{n\to+\infty} p(x_n, x_n) = p(x,x)$.*

Proof. For any sequence (x_n) in (X,p) and for any point $x \in X$, one has that

$$\begin{aligned} x_n \rightarrow x(\text{ properly}) \Longleftrightarrow& \lim_{n\to+\infty} p^m(x_n, x) = 0 \\ \Longleftrightarrow& \lim_{n\to+\infty} (2p(x_n,x) - p(x_n,x_n) - p(x,x)) = 0 \\ \Longleftrightarrow& (\lim_{n\to+\infty} p(x_n,x) - \lim_{n\to+\infty} p(x_n,x_n)) + (\lim_{n\to+\infty} p(x_n,x) - p(x,x)) = 0 \\ \Longleftrightarrow& \lim_{n\to+\infty} p(x_n,x) = \lim_{n\to+\infty} p(x_n,x_n) = p(x,x). \end{aligned}$$

■

In the partial metric space $(\mathbb{R}^-, p)$, the proper limit of the sequence $(-\frac{1}{n})$ is 0 since one has $\lim\limits_{n\to+\infty} p^m(-\frac{1}{n}, 0) = 0$, where p^m is the usual metric induced by p on $\mathbb{R}^-$.

Definition 2.1.4. A sequence (x_n) in a partial metric space (X, p) is a *Cauchy sequence* if it is a *Cauchy sequence* in the induced metric space (X, p^m).

A partial metric space is said to be *complete* if its induced metric space is *complete*.

The partial metric space $(\mathbb{R}^-, p)$ is complete since the induced metric space $(\mathbb{R}^-, p^m)$ such that for any $x, y \in \mathbb{R}^-, p^m(x, y) = |x - y|$ (the usual distance) is complete.

Lemma 2.1.4. *[O'N95] Suppose that (x_n) is a sequence in a partial metric space (X, p). Then (x_n) is a Cauchy sequence if, and only if, $\lim_{n,k\to+\infty} p(x_n, x_k)$ exists.*

Proof. Suppose that (x_n) is a Cauchy sequence in (X, p). Since for any $n, k \in \mathbb{N}$, $|p(x_n, x_n) - p(x_k, x_k)| \leq p^m(x_n, x_k)$, then $\lim_{n,k\to+\infty} p^m(x_n, x_k) = 0$ implies that $(p(x_n, x_n))$ is a Cauchy sequence in $\mathbb{R}^+$, and so $\lim_{n\to+\infty} p(x_n, x_n) \in \mathbb{R}^+$. Then one has that

$$\begin{aligned}
(x_n) \text{ Cauchy sequence} &\iff \lim_{n,k\to+\infty} p^m(x_n, x_k) = 0 \text{ and } \lim_{n\to+\infty} p(x_n, x_n) \in \mathbb{R}^+ \\
&\iff \lim_{n,k\to+\infty} (2p(x_n, x_k) - p(x_n, x_n) - p(x_k, x_k)) = 0 \\
&\quad\;\; \text{and } \lim_{n\to+\infty} p(x_n, x_n) \in \mathbb{R}^+ \\
&\iff \lim_{n,k\to+\infty} (p(x_n, x_k) - p(x_n, x_n)) + \lim_{n,k\to+\infty} (p(x_n, x_k) - p(x_k, x_k)) = 0 \\
&\quad\;\; \text{and } \lim_{n\to+\infty} p(x_n, x_n) \in \mathbb{R}^+ \\
&\iff \lim_{n,k\to+\infty} p(x_n, x_k) \in \mathbb{R}^+.
\end{aligned}$$

■

Remark 2.1. Considering a partial metric space and its induced metric space, we note that every properly convergent sequence is a Cauchy sequence, but the converse is not always true.

Indeed, if one takes $(0, -1, 0, -1, 0, \dots)$ in $\mathbb{R}^-$ with its usual partial metric, then the sequence converges to -1 since

$$\begin{aligned}
\lim_{n\to+\infty} p(x_n, -1) &= \lim_{n\to+\infty} (-\min\{x_n, -1\}) \\
&= \lim_{n\to+\infty} (1) = 1 = p(-1, -1).
\end{aligned}$$

But $\lim_{n,k\to+\infty} p(x_n, x_k)$ does not exist since it can be 0 or -1 since for any $n \in \mathbb{N}$, $x_n = 0$ or $x_n = -1$. That means $(0, -1, 0, -1, 0, \dots)$ is not a Cauchy sequence according to Lemma 2.1.4.

Remark 2.2. In a complete partial metric space, every Cauchy sequence has a proper limit since it has a limit in the induced metric space, which by definition, is the proper limit.

2.2 Completion of a partial metric space

In this section, we construct the completion of an incomplete partial metric space [O'N95] in a manner similar to the case of incomplete metric space [Heu75, Hu66].

Definition 2.2.1. A subset A in a partial metric space (X, p) is said to be *dense* if it is *dense* in the induced metric space (X, p^m).

An *isometry* of partial metric spaces (X, p) and (X', p') is a bijection $f : (X, p) \longrightarrow (X', p')$ such that for any $x, y \in X$, $p(x, y) = p'[f(x), f(y)]$.

If f is an isometry from (X, p) to $(f(X), p')$, then f is called an *isometry into* (X', p').

Considering the induced metric space, one can verify that a subset A is dense in (X, p) if, and only if, every point in X is the proper limit of some sequence in A.

Definition 2.2.2. [O'N95] A *completion* of a partial metric space (X, p) is a complete partial metric space $(\widehat{X}, \widehat{p})$ such that there exists a map $i : X \longrightarrow \widehat{X}$ satisfying

1. i is an isometry into $\widehat{X}$,
2. $i(X)$ is dense in $(\widehat{X}, \widehat{p})$.

Recall that a metric space is a particular case of a partial metric space, we can observe that the standard definition of the completion of a metric space follows from the completion of the partial metric space.

Lemma 2.2.1. *Suppose that (X, p) is a partial metric space with induced metric p^m. Then for any $x, y, z, w \in X$, one has $|p(x, y) - p(z, w)| \leq p^m(x, z) + p^m(y, w)$.*

Proof. For any $x, y, z, w \in X$, since $p^m(x, z) = 2p(x, z) - p(x, x) - p(z, z) \geq p(x, z) - p(z, z)$ and $p^m(y, w) = 2p(y, w) - p(y, y) - p(w, w) \geq p(w, y) - p(w, w)$ we have that

$$\begin{aligned} p(x, y) - p(z, w) \leq& p(x, z) + p(z, y) - p(z, z) - p(z, w) \\ \leq& p^m(x, z) + p(z, w) + p(w, y) - p(w, w) - p(z, w) \\ \leq& p^m(x, z) + p^m(y, w). \end{aligned}$$

Similarly,

$$\begin{aligned} p(z, w) - p(x, y) \leq& p(z, x) + p(x, w) - p(x, x) - p(x, y) \\ \leq& p^m(z, x) + p(x, y) + p(y, w) - p(y, y) - p(x, y) \\ \leq& p^m(z, x) + p(y, w). \end{aligned}$$

■

Lemma 2.2.2. *Consider a partial metric space (X, p) with its induced metric space (X, p^m). Then the space $(\widehat{X}, \widehat{p}^m)$ is a metric space, where $\widehat{X}$ is the set of equivalence classes of Cauchy sequences in (X, p^m) such that $(x_n) \sim (y_n)$ if, and only if, $\lim_{n\to+\infty} p^m(x_n, y_n) = 0$, and for any $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by $(x_n), (y_n)$ respectively, $\widehat{p}^m(\widehat{x}, \widehat{y}) ::= \lim_{n\to+\infty} p^m(x_n, y_n)$.*

Proof. Let be $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by $(x_n), (y_n)$ respectively. If $\widehat{x} = \widehat{y}$, which means that for any $n \in \mathbb{N}, x_n = y_n$, then $p^m(x_n, y_n) = 0$ for any $n \in \mathbb{N}$, hence $\widehat{p}^m(\widehat{x}, \widehat{y}) = 0$. Conversely if $\widehat{p}^m(\widehat{x}, \widehat{y}) = 0 = \lim_{n\to+\infty} p^m(x_n, y_n)$, then $(x_n) \sim (y_n)$ hence $\widehat{x} = \widehat{y}$.

One can show easily that $\widehat{p}^m(\widehat{x}, \widehat{y}) = \widehat{p}^m(\widehat{y}, \widehat{x})$ for any $\widehat{x}, \widehat{y} \in \widehat{X}$.

Let be $\widehat{x}, \widehat{y}, \widehat{z} \in \widehat{X}$ represented by $(x_n), (y_n)$ and (z_n) respectively, and let $\varepsilon > 0$. Then there exists $K > 0$ such that for any $n \geq K$, one has that

$$\begin{aligned} \widehat{p}^m(\widehat{x}, \widehat{y}) \leq & p^m(x_n, y_n) + \varepsilon, \\ p^m(x_n, z_n) \leq & \widehat{p}^m(\widehat{x}, \widehat{z}) + \varepsilon, \\ p^m(z_n, y_n) \leq & \widehat{p}^m(\widehat{z}, \widehat{y}) + \varepsilon. \end{aligned}$$

Thus for $n \geq K$, using the triangle inequality for p^m, one has that

$$\begin{aligned} \widehat{p}^m(\widehat{x}, \widehat{y}) \leq & p^m(x_n) + p^m(z_n, y_n) + \varepsilon \\ \leq & \widehat{p}^m(\widehat{x}, \widehat{z}) + \widehat{p}^m(\widehat{z}, \widehat{y}) + 3\varepsilon. \end{aligned}$$

Since this holds for all $\varepsilon > 0$, then we have that $\widehat{p}^m(\widehat{x}, \widehat{y}) \leq \widehat{p}^m(\widehat{x}, \widehat{z}) + \widehat{p}^m(\widehat{z}, \widehat{y})$.

We then conclude that $\widehat{p}^m$ is a metric, hence $(\widehat{X}, \widehat{p}^m)$ is a metric space. ∎

Lemma 2.2.3. *[Heu75, Hu66] For any two Cauchy sequences $(x_n), (y_n)$ in a given metric space (X, d), the sequence (z_n) such that for any $n \in \mathbb{N}$, $z_n = d(x_n, y_n)$ converges to a real number which will be denoted by $e[(x_n), (y_n)]$, that is $e[(x_n), (y_n)] = \lim_{n\to+\infty} d(x_n, y_n)$.*

Proof. Let $\varepsilon > 0$, then there exists $N \geq 1$ such that for any $i, j \geq N, d(x_i, x_j) < \varepsilon/2$ and $d(y_i, y_j) < \varepsilon/2$. Hence for any $i, j \geq N$, we have that

$$\begin{aligned} |z_i - z_j| = |d(x_i, x_i) - d(x_j, y_j)| \leq & |d(x_i, y_i) - d(x_j, y_i)| + |d(x_j, y_i) - d(x_j, y_j)| \\ \leq & d(x_i, x_j) + d(y_i, y_j) \\ \leq & \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Thus (z_n) is a Cauchy sequence of real numbers and hence converges to a real number which we denote by $e[(x_n), (y_n)]$. ∎

Remark 2.3. Considering Lemma 2.2.3, especially its notations, if (X, p) is a partial metric space, with induced metric space (X, p^m) which has its completion denoted by $(\widehat{X}, \widehat{p}^m)$, then for any $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by $(x_n), (y_n)$ respectively, by definition that one has set to $\widehat{p}^m$, one has $\widehat{p}^m(\widehat{x}, \widehat{y}) = \lim_{n\to+\infty} p^m(x_n, y_n) = e[(x_n), (y_n)]$.

Lemma 2.2.4. *The metric space $(\widehat{X}, \widehat{p}^m)$ where $\widehat{X}$ is the set of equivalence classes of Cauchy sequences in (X, p^m) such that $(x_n) \sim (y_n)$ if, and only if, $\lim_{n\to+\infty} p^m(x_n, y_n) = 0$, is complete.*

Proof. Considering Remark 2.3, we have that for any $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by $(x_n), (y_n)$ respectively, $e[(x_n), (y_n)] = \widehat{p}^m(\widehat{x}, \widehat{y})$.

Now, let $(\widehat{x}_n)$ be a Cauchy sequence in $\widehat{X}$. Since $i(X)$ is dense in $\widehat{X}$, then for any $k \in \mathbb{N}$, there exists $y_k \in X$ such that $\widehat{p}^m(\widehat{x}_k, i(y_k)) < 1/k$ because $\widehat{x}_k$ and $i(y_k)$ are Cauchy sequences in X. Hence one obtains a sequence $\widehat{y} = (y_k)$ in X.

To prove that $\widehat{y}$ is a Cauchy sequence in X, let $\varepsilon > 0$ and since $(\widehat{x}_n)$ is a Cauchy sequence in $\widehat{X}$, then there exists $K > 3/\varepsilon$ such that $\widehat{p}^m(\widehat{x}_r, \widehat{x}_s) < \varepsilon/3$ for any $r, s > K$. Since $i : X \longrightarrow \widehat{X}$ is an isometry, we have that for any $r, s > K$

$$\begin{aligned} p^m(y_r, y_s) =& \widehat{p}^m[i(y_r), i(y_s)] \\ \leq& \widehat{p}^m[i(y_r), \widehat{x}_r] + \widehat{p}^m[\widehat{x}_r, \widehat{x}_s] + \widehat{p}^m[\widehat{x}_s, i(y_s)] \\ \leq& \varepsilon/3 + 1/r + 1/s < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Then (y_k) is a Cauchy sequence in X, hence $\widehat{y} \in \widehat{X}$.

Now let $\varepsilon > 0$. Since (y_k) is a Cauchy sequence in X then there exists $K_1 > 2/\varepsilon$ such that $p^m(x_k, x_n) < \varepsilon/2$ for any $k, n > K_1$. This implies that for any $n > K_1$,
$\widehat{p}^m[(y_k), i(y_n)] = \lim_{k\to+\infty} p^m(y_k, y_n) \leq \varepsilon/2$. Since $(\widehat{x}_n)$ is a Cauchy sequence, for any $n > K_1$

$$\begin{aligned} \widehat{p}^m[(\widehat{x}_n), (y_n)] \leq& \widehat{p}^m[\widehat{x}_n, i(y_n)] + \widehat{p}^m[i(y_n), (y_n)] \\ <& 1/n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves that $\widehat{x}_n \to \widehat{y}$ with $\widehat{y} = (y_n) \in \widehat{X}$. Hence $(\widehat{X}, p^m)$ is complete. ∎

The following theorem is about the completion of a partial metric space which is similar to the case of a metric space with the motivation that it is much easier to work in a complete space rather than in an incomplete one.

Theorem 2.2.5. *[O'N95] Every partial metric space has a completion.*

Proof. Suppose that p^m is the induced metric on X. Let $\widehat{X}$ be the set of equivalence classes of Cauchy sequences in X, where $(x_n) \sim (y_n)$ if, and only if $p^m(x_n, y_n) \to 0$ as $n \to +\infty$.

Then for any $\widehat{x}, \widehat{y} \in \widehat{X}$, represented by (x_n) and (y_n) respectively. The function
$\widehat{p}^m : \widehat{X} \times \widehat{X} \longrightarrow \mathbb{R}^+$ defined by $\widehat{p}^m(\widehat{x}, \widehat{y}) ::= \lim_{n\to+\infty} p^m(x_n, y_n)$ is a metric and $(\widehat{X}, \widehat{p}^m)$ is a completion of (X, p^m) in the metric sense.

Now, for any $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by (x_n) and (y_n) respectively, the function $\widehat{p} : \widehat{X} \times \widehat{X} \longrightarrow \mathbb{R}^+$ defined by $\widehat{p}(\widehat{x}, \widehat{y}) ::= \lim_{n\to+\infty} p(x_n, y_n)$ is a partial metric.

Firstly, let us prove that $\lim_{n\to+\infty} p(x_n, y_n)$ exists. For any $n, k \geq 1$, one has that $0 \leq |p(x_n, y_n) - p(x_k, y_k)| \leq p^m(x_n, x_k) + p^m(y_k, y_n)$ (Lemma 2.2.4) which tends to 0 as $n \to +\infty$. So $(p(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}+$ and then $\lim_{n\to+\infty} p(x_n, y_n)$ exists.

The function $\widehat{p}$ is well defined since if (x'_n) also represents $\widehat{x}$, then one obtains the inequality $0 \leq |p(x'_n, y_n) - p(x_n, y_n)| \leq p^m(x'_n, x_n)$ (Lemma 2.2.4) which tends to 0 as $n \to +\infty$ since they are in the same class, that is $\lim_{n\to+\infty} p(x'_n, y_n) = \lim_{n\to+\infty} p(x_n, y_n)$.

The function $\widehat{p}$ is a partial metric: for any $\widehat{x}, \widehat{y}, \widehat{z} \in \widehat{X}$ represented by $(x_n), (y_n)$ and (z_n) respectively, we have that
P1. $\widehat{p}(\widehat{x}, \widehat{y}) = \lim_{n\to+\infty} p(x_n, x_n) \leq \lim_{n\to+\infty} p(x_n, y_n) = \widehat{p}(\widehat{x}, \widehat{y})$.
P2. Suppose that $\widehat{p}(\widehat{x}, \widehat{x}) = \widehat{p}(\widehat{x}, \widehat{y}) = \widehat{p}(\widehat{y}, \widehat{y})$. Then we have that

$$\begin{aligned} \lim_{n\to+\infty} p(x_n, x_n) = \lim_{n\to+\infty} p(x_n, y_n) = \lim_{n\to+\infty} p(y_n, y_n) &\iff \lim_{n\to+\infty} p^m(x_n, y_n) = 0 \\ &\iff (x_n) \sim (y_n) \\ &\iff \widehat{x} = \widehat{y}. \end{aligned}$$

P3. Since $p(x_n, y_n) = p(y_n, x_n)$, then it follows that $\widehat{p}(\widehat{x}, \widehat{y}) = \widehat{p}(\widehat{y}, \widehat{x})$.
P4. Since p is a partial metric, one has that $p(x_n, z_n) \leq p(x_n, y_n) + p(y_n, z_n) - p(y_n, y_n)$. Taking the limit when $n \to +\infty$, it follows that $\widehat{p}(\widehat{x}, \widehat{z}) \leq \widehat{p}(\widehat{x}, \widehat{y}) + \widehat{p}(\widehat{y}, \widehat{z}) - \widehat{p}(\widehat{y}, \widehat{y})$.

We verify that $\widehat{p}$ induces the metric $\widehat{p}^m$ since, if $\widehat{x}, \widehat{y} \in \widehat{X}$ represented by (x_n) and (y_n) respectively, then $p^m(x_n, y_n) = 2p(x_n, y_n) - p(x_n, x_n) - p(y_n, y_n)$. Taking the limit when $n \to +\infty$, we have that

$$\begin{aligned} \lim_{n\to+\infty} p^m(x_n, x_n) =& 2 \lim_{n\to+\infty} p(x_n, y_n) - \lim_{n\to+\infty} p(x_n, x_n) - \lim_{n\to+\infty} p(y_n, y_n) \\ \widehat{p}^m(\widehat{x}, \widehat{y}) =& 2\widehat{p}(\widehat{x}, \widehat{y}) - \widehat{p}(\widehat{x}, \widehat{x}) - \widehat{p}(\widehat{y}, \widehat{y}). \end{aligned}$$

Since $(\widehat{X}, \widehat{p}^m)$ is complete, then $(\widehat{X}, \widehat{p})$ is complete.

Finally, let us define a function $i : X \longrightarrow \widehat{X}$ by $i(x) = (x, x, x, \dots)$. Let us prove that i is an isometry into $\widehat{X}$ and that $i(X)$ is dense in $(\widehat{X}, \widehat{p})$. First, since $i(X) = \{(x, x, x, \dots) : x \in X\}$, then i is a bijection from X to $i(X)$ by construction and since $p(x, y) = \lim_{n\to+\infty} p(x, y) = \widehat{p}(i(x), i(y))$ for any $x, y \in X$, therefor i is an isometry into $\widehat{X}$.

Let us prove now that $i(X)$ is dense in $(\widehat{X}, \widehat{p}^m)$ which will imply that it is dense in $(\widehat{X}, \widehat{p})$.

Let be $\widehat{x} \in (\widehat{X}, \widehat{p}^m)$ represented by $(x_n), x_n \in X$. We are looking for a sequence $(\widehat{y}_n)$ in $i(X)$ such that $\widehat{p}^m(\widehat{y}_n, \widehat{x}) \to 0$ as $n \to +\infty$. For each $n \in \mathbb{N}$, consider $\widehat{y}_n = (y_{n_1}, y_{n_2}, \dots) = i(x_n)$ where for any $k \in \mathbb{N}$, $y_{n_k} = x_n$. Then we have that

$$\begin{aligned} \widehat{p}(\widehat{y}_n, \widehat{x}) &= \lim_{k\to+\infty} p^m(y_{n_k}, x_k) \\ &= \lim_{k\to+\infty} p^m(x_n, x_k). \end{aligned}$$

Since (x_n) is a Cauchy sequence in X, then $\lim\limits_{n,k\to+\infty} p^m(x_n, x_k) = \lim\limits_{n\to+\infty} \widehat{p}(\widehat{y}_n, \widehat{x}) = 0$. Thus the sequence $(\widehat{y}_n)$ converges to $\widehat{x}$, so $i(X)$ is dense in $\widehat{X}$. ■

Lemma 2.2.6. *If A is a dense subset of a partial metric space (X_1, p_1) and $f : A \longrightarrow X_2$ is an isometry into a complete partial metric space (X_2, p_2), then f extends uniquely to an isometry of X_1 into X_2.*

Proof. If $x \in X_1$, then there exists (x_n) in A properly converging to x, since A is dense in (X_1, p_1). Considering the induced metric spaces, the function $g : X_1 \longrightarrow X_2$ where $g(x)$ is the proper limit of the sequence $(f(x_n))$ in (X_2, p_2) is a well defined extension of f, that is $g|_A = f$.

Indeed, if (a_n) and (x_n) both converge to x, then $\lim\limits_{n\to+\infty} p_1(a_n, x_n) = 0$ and therefore $(f(a_n))$ and $(f(x_n))$ have the same limit.

Now, if $x, y \in X$ and $(x_n), (y_n)$ are sequences in A properly converging to x, y respectively, then one uses the fact that

$$|p_2[f(x_n), f(y_n)] - p_2[g(x), g(y)]| \leq p_2^m[f(x_n), g(x)] + p_2^m[f(y_n), g(y)],$$

since $\lim\limits_{n\to+\infty} p_2^m[f(x_n), g(x)] = 0$, $\lim\limits_{n\to+\infty} p_2^m[f(y_n), g(y)] = 0$, then one has $\lim\limits_{n\to+\infty} p_2[f(x_n), f(y_n)] = p_2[g(x), g(y)]$.

Since f is an isometry, then one has

$$\begin{aligned} p_2[g(x), g(y)] &= \lim_{n\to+\infty} p_2[f(x_n), f(y_n)] \\ &= \lim_{n\to+\infty} p_1(x_n, y_n) \\ &= p_1(x, y). \end{aligned}$$

Suppose that $x, y \in X$ and $(x_n), (y_n)$ are sequences in A properly converging to x, y respectively, and assume that $g(x) = g(y)$. Then $p_2[g(x), g(y)] = 0 = p_1(x, y)$. Since $p_1(x, x) \leq p_1(x, y)$ and $p_1(y, y) \leq p_1(x, y)$, then $p_1(x, x) = p_1(x, y) = p_1(y, y) = 0$, hence $x = y$. Thus g is injective, and consequently an isometry into (X_2, p_2).

To see that g is unique, we note that if g' is any extension of f, and (x_n) is a sequence in A properly converging to $x \in X_1$, then $g'(x)$ is the proper limit of $(g'(x_n))$ which is the same sequence as $(f(x_n))$ since $g'(x_n) = f(x_n)$ as $x_n \in A$ for any $n \in \mathbb{N}$ and $g'|_A = f|_A$. Thus $g'(x) = g(x)$ for any $x \in X_1$, hence $g = g'$. ■

Theorem 2.2.7. *[O'N95] Suppose that (X, p) is a partial metric space with completions $(\widehat{X}, \widehat{p})$ and $(\widehat{X}', \widehat{p}')$ with respect to i and i' respectively. Then there exists a unique isometry $j : \widehat{X} \longrightarrow \widehat{X}'$ such that $j \circ i = i'$.*

Proof. Let $A = i(X) \subset \widehat{X}$, and consider $j : A \longrightarrow \widehat{X}'$ by setting $j(i(x)) = i'(x)$ for any $x \in X$. Then j is injective by construction, and it is an isometry into $(\widehat{X}', \widehat{p}')$.

Suppose that $\widehat{x}' \in \widehat{X}'$, then there exists a sequence $(\widehat{x}'_n) \in i(X)$ such that $\widehat{x}'_n \to \widehat{x}'$. Moreover, for any $n \in \mathbb{N}$, there exists $x_n \in X$ so that $\widehat{x}'_n = i'(x_n)$. If one takes $j[i(x_n)] = i'(x_n)$, then $(j[i(x_n)])$ is a sequence in $j(A)$ such that $j[i(x_n)] \to \widehat{x}'$, hence $j(A)$ is dense in $(\widehat{X}', \widehat{p}')$. Since $(\widehat{X}', \widehat{p}')$ is complete, then j extends uniquely to an isometry of $\widehat{X}$ into $\widehat{X}'$ such that for any $x \in X$, $j[i(x)] = i'(x)$, that is $j \circ i = i'$. ■

Remark 2.4. According to the Theorem 2.2.5 and Theorem 2.2.7, every partial metric space has a unique completion, up to isometry. This property coincide exactly with the case when we have a metric space.

2.3 Banach's fixed point theorem

From the previous chapter, we demonstrated that in a partial metric space (X, p), a sequence (x_n) converges to a point $a \in X$ if, and only if, $\lim_{n,k\to+\infty} p(x_n, x_m) = \lim_{n\to+\infty} p(x_n, a) = p(a, a)$. We will use this notion of convergence to extend *Banach's fixed point theorem* to a complete partial metric space [Val05].

Definition 2.3.1. A map $f : X \longrightarrow X$ of a partial metric space (X, p) into itself is a *contraction* if there exists a constant $0 \leq c < 1$ called the *contraction constant*, such that for any $x, y \in X$,

$$p[f(x), f(y)] \leq cp(x, y).$$

Theorem 2.3.1. *[Val05] The partial metric contraction theorem.*
Let $f : X \longrightarrow X$ be a contraction of a complete partial metric space (X, p) into itself. Then there exists a unique $a \in X$ such that $a = f(a)$ and $p(a, a) = 0$.

Proof. Let be c the contraction constant of f and suppose that $u \in X$, then for any $n, k \in \mathbb{N}$, since $p[f^{n+k}(u), f^{n+k}(u)] \geq 0$, one has

$$\begin{aligned}
p[f^{n+k+1}(u), f^n(u)] \leq & p[f^{n+k+1}(u), f^{n+k}(u)] + p[f^{n+k}(u), f^n(u)] - p[f^{n+k}(u), f^{n+k}(u)] \\
\leq & c^{n+k} p(f(u), u) + p[f^{n+k}(u), f^n(u)] \\
\leq & (c^{n+k} + \cdots + c^n) \times p(f(u), u) + p[f^n(u), f^n(u)] \\
\leq & c^n(1 + c + \cdots + c^k) \times p(f(u), u) + c^n \times p(u, u) \\
\leq & c^n \left[\left(\frac{1 - c^{k+1}}{1 - c} \right) \times p(f(u), u) + p(u, u) \right] \\
\leq & c^n \left[\frac{p(f(u), u)}{1 - c} + p(u, u) \right].
\end{aligned}$$

Thus $(f^n(u))$ is a Cauchy sequence in (X, p) such that $\lim_{n,m\to+\infty} p[f^n(u), f^m(u)] = 0$. As (X, p) is complete, there exists $a \in X$ such that $(f^n(u))$ converges to a and
$p(a, a) = \lim_{n\to+\infty} p(f^n(u), a) = \lim_{n\to+\infty} p[f^n(u), f^n(u)] = 0$.

Now, for any $n \in \mathbb{N}$, we have that

$$\begin{aligned} p(f(a), a) \leq & p[f(a), f^{n+1}(u)] + p(f^{n+1}(u), a) - p[f^{n+1}(u), f^{n+1}(u)] \\ \leq & c \times p(a, f^n(u)) + p(f^{n+1}(u), a). \end{aligned}$$

Taking the limit when $n \to +\infty$, one has $\lim_{n\to+\infty} p(a, f^n(u)) = \lim_{n\to+\infty} p(f^{n+1}(u), a) = 0$ and then $p(f(a), a) = 0$. But since $p(f(a), f(a)) \leq p(f(a), a)$, we have that $p(f(a), f(a)) = p(f(a), a) = p(a, a) = 0$ which implies that $a = f(a)$.

If $b \in X$ is such that $b = f(b)$, then $p(a, b) = p(f(a), f(b)) \leq cp(a, b)$. Since $c < 1$ one has $p(a, b) = 0 = p(b, b) = p(a, a)$. Hence $a = b$, thus the fixed point of f is unique. ■

Theorem 2.3.2. *[Val05] Let $f : X \longrightarrow X$ be a mapping of a complete partial metric space (X, p) into itself such that for any $x, y \in X$,,*

$$p[f(x), f(y)] \leq \phi(p(x, x)),$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is any monotone non-decreasing function with $\lim_{n\to+\infty} \phi^n(t) = 0$ for each fixed $t > 0$. Then f has a unique fixed point a such that $a = f(a)$ and $p(a, a) = 0$.

Proof. First let us note that if $t > 0$, then $\phi(t) < t$ because if $t \leq \phi(t)$, then $\phi(t) \leq \phi(\phi(t))$ and therefore $t \leq \phi^2(t)$. By induction, $t \leq \phi^n(t)$ for $n \geq 1$ and it follows that $t \leq \lim_{n\to+\infty} \phi^n(t) = 0$ a contradiction.

Let us fix $x \in X$. For any $n \in \mathbb{N}$, $p[f^n(x), f^n(x)] \leq \phi^n(p(x, x))$ and $p[f^n(x), f^{n+1}(x)] \leq \phi^n(p[x, f(x)])$. Since $q(x, y) = p(x, y) - p(x, x)$ is a quasi-metric on X such that $\mathcal{T}[q] = \mathcal{T}[p]$, therefor

$$\begin{aligned} q[f^n(x), f^{n+1}(x)] \leq & p[f^n(x), f^{n+1}(x)] + p[f^n(x), f^n(x)] \\ \leq & \phi^n(p[x, f(x)]) + \phi^n(p(x, x)) \\ \leq & 2\phi^n(p[x, f(x)]). \end{aligned}$$

Now let $n, k \in \mathbb{N}$, then

$$\begin{aligned} p[f^n(x), f^{n+k+1}(x)] \leq & p[f^{n+k+1}(x), f^{n+k}(x)] + p[f^{n+k}(x), f^n(x)] - p[f^{n+k}(x), f^{n+k}(x)] \\ \leq & \phi^{n+k}(p[x, f(x)]) + p[f^{n+k}(x), f^n(x)] \\ \leq & (\phi^{n+k} + \cdots + \phi^n)(p[x, f(x)]) + p[f^n(x), f^n(x)]. \end{aligned}$$

Thus one has that

$$p[f^n(x), f^{n+k+1}(x)] \leq (\phi^{n+k} + \cdots + \phi^n)(p[x, f(x)]) + \phi^n(p(x, x)),$$

where the term in the right tends to 0 as $n, k \to +\infty$.

Hence $(f^n(x))$ is a Cauchy sequence in (X, p) such that $\lim_{n,k\to+\infty} p[f^n(x), f^k(x)] = 0$, and since (X, p) is complete, there exists $a \in X$ satisfying $p(a, a) = \lim_{n\to+\infty} p[f^n(x), a] = \lim_{n,k\to+\infty} p[f^n(x), f^k(x)] = 0.$

Moreover, for any $n \in \mathbb{N}$,

$$\begin{aligned} p[f(a), a] \leq & p[f(a), f^{n+1}(x)] + p[f^{n+1}(x), a] - p[f^{n+1}(x), f^{n+1}(x)] \\ \leq & \phi(p[a, f^n(x)]) + p[f^{n+1}(x), a] \\ \leq & p[a, f^n(a)] + p[f^{n+1}(x), a]. \end{aligned}$$

Taking the limit when $n \to +\infty$, one has that $\lim_{n\to+\infty} p[a, f^n(x)] = \lim_{n\to+\infty} p[f^{n+1}(x), a] = 0$, hence $p(f(a), a) = 0$. Since $p[f(a), f(a)] \leq p[f(a), a]$, therefor $p[f(a), f(a)] = p[f(a), a] = p(a, a) = 0$, thus $a = f(a)$.

Now let us prove the uniqueness. If $b \in X$ such that $b = f(b)$, then we have that for any $a, b \in X$, $p(a, b) = p[f(a), f(b)] \leq \phi(p(a, b))$.

If $p(a, b) \neq 0$, then $\phi(p(a, b)) < p(a, b)$ and the relation would not hold. Thus $p(a, b) = 0$ and therefor $p(b, b) = p(a, b) = p(a, a) = 0$ which implies that $a = b$, hence the fixed point of f is unique. ■

Remark 2.5. One could observe that the first theorem follows as a special case of the preceding result if one chooses $\phi(t) = ct$ with $t > 0$ and $0 \leq c < 1$.

Banach's fixed point theorem is usually used to prove the *Picard-Lindelöf* theorem about the existence and uniqueness of solutions of some initial value problem of the form $y' = f(x, y)$, with $y(x_0) = y_0$ consisting of finding a continuously differentiable function y. The proof consists of transforming the initial value problem into the equivalent form of *Volterra* integral equation

$$u(x) = u_0 + \int_{x_0}^{x} f(\varepsilon, u(\varepsilon))\mathrm{d}\varepsilon.$$

The resolution of the Volterra integral equation needs to employ the Banach's fixed point theorem in a complete space.

Another examples of application of Banach's fixed point theorem is in computer science [Bre98]. The uniqueness of fixed points is useful for comparing *operational* and *denotational semantic models*. To show that semantics models are equal, the models are considered to be in a complete space and consider a contractive function from the partial metric space to itself. Then one proves that the models are fixed points of the function introduced, and by the uniqueness of fixed point, one concludes that the models are equal.

3. Semivaluations and partial metrics

In this chapter, we present the correspondence between co-weightable quasi-metrics and partial metrics, then we turn to the relation between partial metrics and semivaluations [Sch04, RS05].

3.1 Partial metrics and co-weightable quasi-metrics

Definition 3.1.1. [Sch04] The *conjugate* q^{-1} of a quasi-metric q on a non-empty set X is defined to be the function $q : X \times X \longrightarrow \mathbb{R}^+$ such that $q^{-1}(x, y) = q(y, x)$ for any $x, y \in X$, which is again a quasi-metric.

A *co-weighted* quasi-metric is a quasi-metric such that its conjugate has a weighting function, called the *co-weighting function*. A *co-weighted quasi-metric space* is a triple (X, q, ω) consisting of a quasi-metric space (X, q) and a co-weighted quasi-metric (q, ω).

Theorem 3.1.1. *For each partial metric $p : X \times X \longrightarrow \mathbb{R}^+$ on a non-empty set X, the function $q : X \times X \longrightarrow \mathbb{R}^+$ defined by $q(y, x) ::= p(x, y) - p(x, x)$ for any $x, y \in X$ is a co-weighted quasi-metric with co-weighting function $\omega : X \longrightarrow \mathbb{R}^+$, such that for any $x \in X, \omega(x) = p(x, x)$. Moreover, $\mathcal{T}[q^{-1}] = \mathcal{T}[p]$ and for any $x, y \in X$, $x \sqsubseteq_{q^{-1}} y$ if, and only if, $x \sqsubseteq_p y$.*

Proof. Let us first show that q is a quasi-metric. This is clear that for any $x, y \in X, q(x, x) = 0$ and $q(x, y) \geq 0$. Suppose that $q(y, x) = q(x, y) = 0$, then $p(x, y) - p(x, x) = p(y, x) - p(y, y) = 0$ hence $p(x, y) = p(x, x) = p(y, y) = 0$ which implies that $x = y$. Now, since for any $x, y, z \in X$, $p(x, z) - p(x, x) \leq p(x, y) - p(x, x) + p(y, z) - p(y, y)$ then $q(z, x) \leq q(z, y) + q(y, x)$. Hence q is a quasi-metric on X.

For any $x, y \in X$, since $q(y, x) = p(x, y) - \omega(x)$ and $q(x, y) = p(y, x) - \omega(y)$, taking into account that $p(x, y) = p(y, x)$, it follows that $q^{-1}(x, y) + \omega(x) = q^{-1}(y, x) + \omega(y)$, thus (q, ω) is a co-weighted quasi-metric on X.

Now, let us prove that $\mathcal{T}[q^{-1}] = \mathcal{T}[p]$. For any $x \in X$ and $\varepsilon > 0$ such that $x \in B^p_\varepsilon(x)$, if $y \in B^p_\varepsilon(x)$ then $q(y, x) = p(x, y) - p(x, x) < \varepsilon - p(x, x)$ so that $y \in B^{q^{-1}}_{\varepsilon - p(x,x)}(x)$. Conversely, if $y \in B^{q^{-1}}_{\varepsilon - p(x,x)}(x)$, then $y \in B^p_\varepsilon(x)$, hence for any $x \in X$ and $\varepsilon > p(x, x)$, $B^p_\varepsilon(x) = B^{q^{-1}}_{\varepsilon - p(x,x)}(x)$.

For any $x \in X$ and $0 < \varepsilon \leq p(x, x)$, we have that $B^p_\varepsilon(x) = \emptyset$.

For any $x \in X$ and $\varepsilon > 0$, if $y \in X, y \in B^{q^{-1}}_\varepsilon(x)$, then $p(x, y) - p(x, x) < \varepsilon$ which implies that $y \in B^p_{\varepsilon + p(x,x)}(x)$. Conversely, if $y \in B^p_{\varepsilon + p(x,x)}(x, x)$, it is then clear that $q^{-1}(x, y) < \varepsilon$ so that $y \in B^{q^{-1}}_\varepsilon(x)$. Hence for any $x \in X$ and $\varepsilon > 0$, $B^{q^{-1}}_\varepsilon(x) = B^p_{\varepsilon + p(x,x)}(x, x)$.

Finally, for any $x, y \in X$, $p(x, x) = p(x, y)$ if, and only if, $q^{-1}(x, y) = 0$. Thus $x \sqsubseteq_p y$ if, and only if, $x \sqsubseteq_{q^{-1}} y$. ■

Theorem 3.1.2. *For each co-weighted quasi-metric* (q, ω) *on a non-empty set* X*, the function* $p : X \times X \longrightarrow \mathbb{R}^+$ *defined by* $p(x, y) ::= q(y, x) + \omega(x) = q^{-1}(x, y) + \omega(x)$ *for any* $x, y \in X$ *is a partial metric on* X *such that for any* $x \in X, p(x, x) = \omega(x)$*. Moreover,* $\mathcal{T}[p] = \mathcal{T}[q^{-1}]$ *and for any* $x, y \in X$*,* $x \sqsubseteq_p y$ *if, and only if,* $x \sqsubseteq_{q^{-1}} y$*.*

Proof. It is clear that for any $x \in X$, one has that $p(x, x) = \omega(x)$.
P1. For any $x, y \in X, 0 \leq q(y, x)$, then $\omega(x) \leq q(y, x) + \omega(x)$. Since $p(x, x) = \omega(x)$ for any $x \in X$, we have that $p(x, x) \leq p(x, y)$.
P2. Suppose that for any $x, y \in X$, $p(x, x) = p(x, y) = p(y, y)$, then $\omega(x) = q(y, x) + \omega(x) = \omega(y)$. Since (q, ω) is a co-weighted quasi-metric, one has $\omega(x) = q(x, y) + \omega(y) = \omega(y)$. Thus $q(x, y) = q(y, x) = 0$ and hence $x = y$. The reverse is obvious.
P3. Since (q, ω) is a co-weighted quasi-metric, then for any $x, y \in X$, $q(y, x) + \omega(x) = q(x, y) + \omega(y)$ which implies $p(x, y) = p(y, x)$.
P4. For any $x, y, z \in X$ we have that

$$\begin{aligned} q(x, z) &\leq q(x, y) + q(y, z), \\ q(x, z) + \omega(z) &\leq q(x, y) + \omega(y) + q(y, z) + \omega(z) - \omega(y), \\ p(x, z) &\leq p(x, z) + p(y, z) - p(y, y). \end{aligned}$$

Thus p is a partial metric on X.

Let us prove that $\mathcal{T}[p] = \mathcal{T}[q^{-1}]$. For any $x \in X$ and $\varepsilon > 0$ such that $x \in B^p_\varepsilon(x)$, if $y \in X, y \in B^p_\varepsilon(x)$, that is $p(x, y) < \varepsilon$, then $q(y, x) < \varepsilon - \omega(x)$ and hence $y \in B^{q^{-1}}_{\varepsilon-\omega(x)}(x)$. Conversely, if $y \in B^{q^{-1}}_{\varepsilon-\omega(x)}(x)$, then $p(x, y) < \varepsilon$ so that $y \in B^p_\varepsilon(x)$. Hence $B^p_\varepsilon(x) = B^{q^{-1}}_{\varepsilon-\omega(x)}(x)$.

For any $x \in X$ and $0 < \varepsilon \leq \omega(x)$, $B^p_\varepsilon(x) = \emptyset$.

For any $x \in X$ and $\varepsilon > 0$, if $y \in X, y \in B^{q^{-1}}_\varepsilon(x)$, that is $q(y, x) < \varepsilon$, then $p(x, y) < \varepsilon + \omega(x)$ hence $y \in B^p_{\varepsilon+\omega(x)}(x)$. Conversely, if $y \in B^p_{\varepsilon+\omega(x)}(x)$, then $q(y, x) < \varepsilon$ so that $y \in B^{q^{-1}}_\varepsilon(x)$. Hence $B^{q^{-1}}_\varepsilon(x) = B^p_{\varepsilon+\omega(x)}(x)$.

Finally, for any $x, y \in X$, one has $p(x, x) = p(x, y)$ if, and only if, $q(y, x) = 0$. Therefor $x \sqsubseteq_p y$ if, and only if, $x \sqsubseteq_{q^{-1}} y$. ■

Note that the demonstrations are similar to the ones of correspondence between partial metrics and weightable quasi-metrics in Chapter 1.

3.2 Semivaluations

Semivaluations and partial metrics occur in computer science for the study of *quantitative domain theory* [Sch04, RS03]. In this section, we present the bijection between them [Sch04] through the notion of co-weightable quasi-metric.

Definition 3.2.1. A *join (meet) semilattice* is a partial order $(X, \leq)$ such that every two elements $x, y \in X$ have a *supremum* $x \sqcup y$(*infimum* $x \sqcap y$) in X. A *lattice* is a partial order which is both a join and a meet semilattice.

A function $f : X \longrightarrow \mathbb{R}^+$ is *fading* if $\inf_{x \in X} f(x) = 0$.

Recall that the associated partial order $\leq_d$ of a quasi-metric d is defined by $x \leq_d y$ if, and only if, $d(x, y) = 0$ for any $x, y \in X$. The converse is also true, given a partial order $\leq$ on a non-empty set X, there exists a quasi-metric q such that $\leq = \leq_q$ given for any $x, y \in X$ by $d(x, y) = 0$ if $x \leq y$, 0 otherwise.

A quasi-metric space (X, d) is simply said to be a *join (meet) semilattice* (X, d) if the associated partial order $(X, \leq_d)$ is a *join (meet) semilattice*. A weighted quasi-metric space is *of fading weight* if its weighting function is fading.

Definition 3.2.2. A *meet valuation* is a function $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ of a meet semilattice $(X, \preccurlyeq)$ into $(\mathbb{R}^+, \leq)$ such that, for any $x, y, z \in X$, $f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)$, f is a *meet co-valuation* if for any $x, y, z \in X$, $f(x \sqcap z) \leq f(x \sqcap y) + f(y \sqcap z) - f(y)$.

Definition 3.2.3. A *join valuation* is a function $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ of a join semilattice $(X, \preccurlyeq)$ into $(\mathbb{R}^+, \leq)$ such that, for any $x, y, z \in X$, $f(x \sqcup z) \leq f(x \sqcup y) + f(y \sqcup z) - f(y)$, f is a *join co-valuation* if for any $x, y, z \in X$, $f(x \sqcup z) \geq f(x \sqcup y) + f(y \sqcup z) - f(y)$.

Definition 3.2.4. [Sch04] A function is a *semivaluation* if it is either a join valuation or a meet valuation. A *join (meet) valuation space* is a join (meet) semilattice equipped with a join (meet) valuation. A *semivaluation space* is a semilattice equipped with a semivaluation.

Lemma 3.2.1. *Semivaluations are increasing.*

Proof. Assume that $(X, \preccurlyeq)$ is a join semilattice and $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ a join valuation. Let $z = x$ and assume that $x \preccurlyeq y$. Since $f(x \sqcup z) \leq f(x \sqcup y) + f(y \sqcup z) - f(y)$, one obtains that $f(x) \leq f(y)$ since $f(x \sqcup y) = f(y \sqcup z) = f(y)$.

Assume that $(X, \preccurlyeq)$ is a meet semilattice and $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ a meet valuation. Let $z = x$ and assume that $x \preccurlyeq y$. Since $f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)$, one obtains that $f(y) \geq f(x)$ since $f(x \sqcap y) = f(y \sqcap z) = f(x)$. ■

Definition 3.2.5. A join semilattice (X, d) is *invariant* if for any $x, y, z \in X$, one has $d(x \sqcup z, y \sqcup z) \leq d(x, y)$. A meet semilattice (X, d) is *invariant* if for any $x, y, z \in X$, one has $d(x \sqcap z, y \sqcap z) \leq d(x, y)$.

Lemma 3.2.2. *[Sch04] Monotonicity Lemma.*
If (X, d) is a quasi-metric space, then for any $x, y, z \in X$,

$(x' \leq_d x$ and $y \leq_d y')$ implies that $d(x', y') \leq d(x, y)$.

Proof. We have that $x' \leq_d x \iff d(x', x) = 0$ and $y \leq_d y' \iff d(y, y') = 0$. Thus by the triangle inequality, $d(x', y') \leq d(x', x) + d(x, y) + d(y, y')$ and hence the result. ■

Lemma 3.2.3. *[Sch04] A join semilattice (X, d) is invariant if, and only if, for any $x, y \in X$, one has $d(x \sqcup y, y) = d(x, y)$. A meet semilattice (X, d) is invariant if, and only if, for any $x, y \in X$, one has $d(x, x \sqcap y) = d(x, y)$.*

Proof. If (X, d) is a join semilattice such that for any $x, y \in X$, $d(x \sqcup y, y) = d(x, y)$, then for any $x, y, z \in X$, $d(x \sqcup z, y \sqcup z) = d(x \sqcup y \sqcup z, y \sqcup z) = d(x \sqcup (y \sqcup z), (y \sqcup z)) = d(x, y \sqcup z) \leq d(x, y)$ where the last inequality follows by Lemma 3.2.2.

Conversely, assume that (X, d) is an invariant join semilattice. One then has for any $x, y \in X$, $d(x \sqcup y, y) = d(x \sqcup y, y \sqcup y) \leq d(x, y)$ and by Lemma 3.2.2, $d(x, y) \leq d(x \sqcup y, y)$ and hence the equality $d(x, y) = d(x \sqcup y, y)$.

Similarly, if (X, d) is a meet semilattice such that for any $x, y \in X$, $d(x, x \sqcap y) = d(x, y)$, then for any $x, y, z \in X$, $d(x \sqcap z, y \sqcap z) = d(x \sqcap z, x \sqcap y \sqcap z) = d((x \sqcup z), (x \sqcup z) \sqcup y) = d(x \sqcap z, y) \leq d(x, y)$ where the last inequality follows by Lemma 3.2.2.

Conversely, assume that (X, d) is an invariant meet semilattice. We then have that for any $x, y \in X$, $d(x, x \sqcap y) = d(x \sqcap x, x \sqcap y) \leq d(x, y)$ and by Lemma 3.2.2, $d(x, y) \leq d(x, x \sqcap y)$ and hence the equality $d(x, y) = d(x, x \sqcap y)$. ■

Definition 3.2.6. A *partial metric join (meet) semilattice* is a partial metric space which is a join (meet) semilattice for its associated order.

Lemma 3.2.4. *[RS03] Let (X, p) be a partial metric space and (X, q) its induced quasi-metric space. Then $(X, \sqsubseteq_p)$ is a join (meet) semilattice if, and only if, $(X, \sqsubseteq_q)$ is a join (meet) semilattice.*

Proof. One has already proved that $\sqsubseteq_p = \sqsubseteq_q$, thus the result. ■

Definition 3.2.7. A partial metric join (meet) semilattice is *invariant* if its induced quasi-metric join (meet) semilattice is invariant.

Lemma 3.2.5. *The weighting functions of a weightable quasi-metric space are strictly decreasing. The weighting functions are exactly the functions $f + c$ where $c \geq 0$ and where f is the unique fading weighting of the space.*

Proof. Suppose that w is a weighting function of a quasi-metric space (X, d), then the function $f_w = w - L$, where $L = \inf_{x \in X} w(x)$ is a fading weighting function of q. It is also remarked that any weighting function w is of the form $f_w + c$, and one verifies that $f_w + c$ is a weighting function for (X, d), since for any $x, y \in X, d(x, y) + f_w(x) + c = d(y, x) + f_w(y) + c$.

Now suppose that w is a weighting function of (X, d), then for any $x, y \in X$ such that $x <_d y$, one has $d(x, y) = 0$ and $d(y, x) > 0$. Since $d(x, y) + w(x) = d(y, x) + w(y)$, then $w(x) - w(y) = d(y, x) > 0$ and thus $w(x) > w(y)$, hence w is decreasing.

Suppose that f_1 and f_2 are fading weighting functions of a quasi-metric d. Then for any $x, y \in X$, one has $d(x, y) + f_1(x) = d(y, x) + f_1(y)$ and $d(x, y) + f_2(x) = d(y, x) + f_2(y)$. By subtracting each members, one has that $f_1(x) - f_2(x) = f_1(y) - f_2(y)$. Hence there exists a constant c such that $f_1 - f_2 = c$. Since the functions are fading, this implies that the constant must be 0. That is f_1 and f_2 coincide. ■

The following theorems give correspondence between invariant partial metric join (meet) semilattice and fading strictly increasing join (meet) valuations.

Theorem 3.2.6. *[Sch04] For every join semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant co-weighted quasi-metrics d on X such that $\sqsubseteq_d = \preccurlyeq$ and fading strictly increasing join valuations $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$. The map $f \longmapsto d_f$ such that $d_f(x, y) = f(x \sqcup y) - f(y)$. The inverse is the function which to each invariant co-weighted join semilattice (X, q) associates its unique fading co-weighting function.*

Proof. Let us prove first that d_f is a co-weighted quasi-metric. For any $x, y \in X$, $d_f(x, y) = f(x \sqcup y) - f(y)$ with $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ a fading strictly increasing join valuation. Then if $x, y \in X$ such that $d_f(x, y) = 0$ and $d_f(y, x) = 0$, one has that $f(x \sqcup y) = f(y)$ and $f(y \sqcup x) = f(x)$. Since $x \sqcup y = y \sqcup x$ and f strictly increasing, necessarily $x = y$. Besides, for any $x, \in X$, $d_f(x, x) = f(x \sqcup x) - f(x) = 0$.

In addition, for any $x, y, z \in X$, we have that $d_f(x, y) + d_f(y, z) = f(x \sqcup y) - f(y) + f(y \sqcup z) - f(z) \geq f(x \sqcup z) - f(z) = d_f(x, z)$, where the last inequality follows from the fact that f is a join valuation. Hence d_f is a quasi-metric.

For any $x, y \in X$, one has that $d_f(x, y) + f(y) = f(x \sqcup y) = f(y \sqcup x) = d_f(y, x) + f(x)$, hence f is a co-weighting function of d_f.

Now, let us verify that d_f coincides with $\preccurlyeq$. For any $x, y \in X$, we have that $x \leq_{d_f} y \Longleftrightarrow d_f(x, y) = 0 \Longleftrightarrow f(x \sqcup y) - f(y) = 0 \Longleftrightarrow y = x \sqcup y \Longleftrightarrow x \preccurlyeq y$.

To verify that the quasi-metric space (X, d_f) is invariant, one notes that for any $x, y \in X$, $d_f(x, y) = f(x \sqcup y) - f(y) = f((x \sqcup y) \sqcup y) - f(y) = d_f(x \sqcup y, y)$.

Conversely, suppose that (X, d) is an invariant co-weighted join semilattice. Let f_d be the unique fading co-weighting function of (X, d), $f_d : (X, \leq_d) \longrightarrow (\mathbb{R}^+, \leq)$. Suppose that for $x, y \in X, x <_d y$, then $d(x, y) = 0$ and $d(y, x) > 0$. Since $d(y, x) + f_d(x) = d(x, y) + f_d(y)$, then $f_d(y) - f_d(x) = d(y, x) > 0$, hence f_d is strictly increasing.

Let us prove that f_d is a join valuation. Since (X, d) is invariant with co-weighting function f_d, for any $x, y \in X$, $d(x, y) = d(x \sqcup y, y)$ and since $d(x \sqcup y, y) + f_d(y) = d(y, x \sqcup y) + f_d(x \sqcup y)$, we have that $d(x, y) = f_d(x \sqcup y) - f_d(y) = d(x \sqcup y, y)$. Since for any $x, y, z \in X$, $d(x \sqcup z, y \sqcup z) \leq d(x, y)$, then $f_d(x \sqcup y \sqcup z) - f_d(y \sqcup z) \leq f_d(x \sqcup y) - f_d(y)$, and hence $f_d(x \sqcup z) \leq f_d(x \sqcup y \sqcup z) \leq f_d(x \sqcup y) + f_d(y \sqcup z) - f_d(y)$ since f_d is increasing. Thus f_d is a join valuation.

Finally, let us prove that the correspondence obtained is a bijection, that is the maps $f \longmapsto d_f$ and $d \longmapsto f_d$ are inverse to each other.

a. To verify that $d_{f_d} = d$, one remarks that for any $x, y \in X$, one has that $d(x, y) = d(x \sqcup y, y) = f_d(x \sqcup y) - f_d(y) = d_{f_d}(x, y)$ where the last equality is the definition of d_{f_d} and the rest was already shown previously.

b. To verify that $f_{d_f} = f$, one notes also that f is a co-weighting function of d_f, and by the uniqueness of the fading weightings, one has $f = f_{d_f}$. ∎

Theorem 3.2.7. *[Sch04] For every meet semilattice $(X, \preccurlyeq)$, there exists a bijection between invariant co-weighted quasi-metrics q on X with $\sqsubseteq_q = \preccurlyeq$ and fading strictly increasing meet valuations $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$. The map $f \longmapsto d_f$ such that $d_f(x, y) = f(x) - f(x \sqcap y)$.*

The inverse is the function which to each invariant co-weighted quasi-metric space (X, q) with $\sqsubseteq_q = \preccurlyeq$, associates its unique fading co-weighting function.

Proof. Let us prove first that d_f is a co-weighted quasi-metric. For any $x, y \in X$, one has $d_f(x, y) = f(x) - f(x \sqcap y)$ with $f : (X, \preccurlyeq) \longrightarrow (\mathbb{R}^+, \leq)$ a fading strictly increasing meet valuation. Then if $x, y \in X$ such that $d_f(x, y) = 0$ and $d_f(y, x) = 0$, we have that $f(x \sqcap y) = f(x)$ and that $f(y \sqcap x) = f(y)$. Since $x \sqcap y = y \sqcap x$ and f is strictly increasing, necessarily $x = y$. On the other hand, for any $x \in X$, we have that $d_f(x, x) = f(x) - f(x \sqcap x) = 0$.

Besides, for any $x, y, z \in X$, one has that $f(x \sqcap z) \geq f(x \sqcap y) + f(y \sqcap z) - f(y)$ since f is a meet valuation. Hence $-f(x \sqcap z) \leq -f(x \sqcap y) - f(y \sqcap z) + f(y)$, so that $f(x) - f(x \sqcap z) \leq f(x) - f(x \sqcap y) - f(y \sqcap z) + f(y)$, hence $d_f(x \sqcap z) \leq d_f(x, y) + d_f(y, z)$, thus d_f is a quasi-metric.

For any $x, y \in X$, one has that $d_f(x, y) + f(y) = f(x) + f(y) - f(x \sqcap y) = f(y) + f(x) - f(y \sqcap x) = d_f(y, x) + f(x)$, hence f is a co-weighting function of d_f.

Now, let us verify that d_f coincides with $\preccurlyeq$. For any $x, y \in X$, one has that $x \leq_{d_f} y \Longleftrightarrow d_f(x, y) = 0 \Longleftrightarrow f(x) - f(x \sqcap y) = 0 \Longleftrightarrow x = x \sqcap y \Longleftrightarrow x \preccurlyeq y$.

To verify that the quasi-metric space (X, d_f) is invariant, one notes that for any $x, y \in X$, $d_f(x, y) = f(x) - f(x \sqcap y) = f(x) - f(x \sqcap (x \sqcap y)) = d_f(x, x \sqcap y)$.

Conversely, suppose that (X, d) is an invariant co-weighted meet semilattice. Let f_d be the unique fading co-weighting function of (X, d), $f_d : (X, \leq_d) \longrightarrow (\mathbb{R}^+, \leq)$. Suppose that for $x, y \in X, x <_d y$, then $d(x, y) = 0$ and $d(y, x) > 0$. Since $d(y, x) + f_d(x) = d(x, y) + f_d(y)$, we have that $f_d(y) - f_d(x) = d(y, x) > 0$, hence f_d is strictly increasing.

Let us prove that f_d is a meet valuation. Since (X, d) is invariant with co-weighting function f_d, for any $x, y \in X$, we have that $d(x, y) = d(x, x \sqcap y)$ and since $d(x, x \sqcap y) + f_d(x \sqcap y) = d(x \sqcap y, x) + f_d(x)$, we then have that $d(x, x \sqcap y) = f_d(x) - f_d(x \sqcap y) = d(x, y)$. Since for any $x, y, z \in X$, $d(x \sqcap z, y \sqcap z) \leq d(x, y)$, then $f_d(x \sqcap z) - f_d(x \sqcap y \sqcap z) \leq f_d(x) - f_d(x \sqcap y)$, therefor $f_d(z \sqcap z) \geq f_d(y \sqcap x \sqcap z) \geq f_d(z \sqcap x) + f_d(x \sqcap y) - f_d(x)$ since f_d is increasing. Thus f_d is a meet valuation.

Finally, let us prove that the correspondence obtained is a bijection, that is the maps $f \longmapsto d_f$ and $d \longmapsto f_d$ are inverse to each other.

a. To verify that $d_{f_d} = d$, one remarks that for any $x, y \in X$, one has that $d(x, y) = d(x, x \sqcap y) = f_d(x) - f_d(x \sqcap y) = d_{f_d}(x, y)$ where the last equality is the definition of d_{f_d} and the rest was already shown previously.

b. To verify that $f_{d_f} = f$, one notes also that f is a weighting function of d_f, and by the uniqueness of the fading weightings, one has $f = f_{d_f}$. ∎

Remark 3.1. According to the correspondence between co-weightable quasi-metrics and partial metrics, and considering the two last Theorems, we obtain that the bijection between invariant partial metric semilattice and fading strictly increasing semivaluations.

4. Examples and applications

Recall that a partial metric space induces a natural partial order, let us consider other examples of partial metric spaces [O'N95, RS05], which have applications in computer science used for the study of semantics and domains, considering partially ordered set [Mat94, But02, Eda97].

Example 4.1. Let $P\omega$ denote the power set of the natural numbers $\omega = \mathbb{N}^* = \mathbb{N}\backslash\{0\}$ different of zero, with the subset ordering. The function $p : P\omega \times P\omega \longrightarrow [0, 1]$ such that

$$p(x, y) = 1 - \sum_{n\in x\cap y} 2^{-n} \text{ for any } x, y \in P\omega$$

is a partial metric on $P\omega$.

Proof. First, for any $x, y \in P\omega$, $p(x, y) \geq 1 - \sum_{n\in\mathbb{N}\backslash\{0\}} 2^{-n} = 0$. Since $\sum_{n=1}^{N} 2^{-n} = 1 - (\frac{1}{2})^N \leq 1$ for any $N \geq 1$, we have that $\sum_{n\in\mathbb{N}\backslash\{0\}} 2^{-n} = 1$.

P1. For any $x, y \in P\omega$, $x \cap y \subset x$ and so $\sum_{n\in x\cap y} 2^{-n} \leq \sum_{n\in x} 2^{-n}$, hence $1 - \sum_{n\in x\cap y} 2^{-n} \geq 1 - \sum_{n\in x} 2^{-n}$, consequently $p(x, y) \geq p(x, x)$.

P2. Suppose that $p(x, x) = p(x, y) = p(y, y)$, then

$$1 - \sum_{n\in x} 2^{-n} = 1 - \sum_{n\in x\cap y} 2^{-n} = 1 - \sum_{n\in y} 2^{-n}.$$

- $x \subseteq y$: if one has $n \in x$, then $n \in y$, otherwise if $n \notin y$, then $p(x, y) > p(x, x)$.
- Similarly, we can prove that $y \subseteq x$. Thus $x = y$.

P3. Since $x \cap y = y \cap x$ for any $x, y \in P\omega$, it follows that $p(x, y) = p(y, x)$ for any $x, y \in P\omega$.

P4. Let $x, y, z \in P\omega$. Using the property of sets, one has that

$$(x \cap y) \cup (y \cap z) = (x \cap y \cap z) \cup (y \cap (x \cup z)).$$

Thus one has

$$\sum_{n\in x\cap y} 2^{-n} + \sum_{n\in y\cap z} 2^{-n} = \sum_{n\in x\cap z\cap y} 2^{-n} + \sum_{n\in y\cap(x\cup z)} 2^{-n}.$$

Consequently, $p(x, y) + p(y, z) = p(x \cap z, y) + p(y, x \cup z) \geq p(x, z) + p(y, y)$.

Hence $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$ for any $x, y, z \in P\omega$. ∎

The induced metric from p on $P\omega$ is d defined by $d(x,y) = \sum_{n \in x \triangle y} 2^{-n}$ for any $x, y \in P\omega$, where for any $x, y \in P\omega$, $x \triangle y = (x \backslash y) \cup (y \backslash x)$.

Indeed, since $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ and using the properties of sets, we have that $x \cup y = ((x \backslash y) \cup (y \backslash x)) \cup ((x \cap y) \cup (x \cap y))$. We then have that

$$\sum_{n \in x} 2^{-n} + \sum_{n \in y} 2^{-n} = \sum_{n \in x \triangle y} 2^{-n} + \sum_{n \in x \cap y} 2^{-n}$$

and it follows that $2p(x,y) - p(x,x) - p(y,y) = \sum_{n \in x \triangle y} 2^{-n}$.

Proposition 4.0.8. *$P\omega$ with its usual partial metric is complete.*

Proof. Suppose that d is the induced metric on $P\omega$ and let (x_n) be a Cauchy sequence in $P\omega, d$. Then for any $k \geq 1$, there exists $N_k \geq 1$ such that for any $n, m \geq N_k$, $\sum_{i \in x_n \triangle x_m} 2^{-i} = d(x_n, x_m) < 2^{-k}$. In particular, for any $n, m \geq N_k$, $k \in x_n$ if, and only if, $k \in x_m$ because if $k \in x_n$ but $k \notin x_m$, then $k \in x_n \triangle x_m$ and the relation $\sum_{i \in x_n \triangle x_m} 2^{-i} < 2^{-k}$ does not hold.

Consider $x = \{k \in \mathbb{N} \backslash \{0\} : k \in x_{N_k}\} \in P\omega$. Then let us prove that (x_n) converges to x. Let be $\varepsilon > 0$, there exists $k \geq 1$ such that $0 < 2^{-k} < \varepsilon$. For any $n \geq \max\{N_i : 1 \geq i \geq k\}$ and $1 \geq i \geq k$, we have that $i \in x_n \iff i \in x_{N_i} \iff i \in x$.
Thus for any $i \in x_n \triangle x$, one must have that $i > k$.

Hence for any $n \geq \max\{N_i : 1 \geq i \geq k\}$, $d(x_n, x) = \sum_{i \in x_n \triangle x} 2^{-i} \geq \sum_{i=k+1}^{+\infty} 2^{-i} = 2^{-k} < \varepsilon$.

Thus (x_n) converges to x in $(P\omega, d)$, and hence the result. ∎

Proposition 4.0.9. *$(P\omega, p)$ is an invariant partial metric meet semilattice.*

Proof. Let us denote $(P\omega, \leq_p)$ the partial order induced by $P\omega, p$ defined for any $x, y \in P\omega$ by $x \leq_p y \iff p(x,x) = p(x,y) \iff \sum_{n \in x} 2^{-n} = \sum_{n \in x \cap y} 2^{-n} \iff x \subseteq y$.

Then $(P\omega, \leq_p)$ is a semilattice, for any $x, y \in P\omega$, one has $x \sqcap y = x \cap y$. The induced quasi-metric is q defined for any $x, y \in P\omega$ by $q(x,y) = p(x,y) - p(x,x)$. Since for any $x, y \in P\omega$, one has that $x \sqcap y = x \sqcap (x \sqcap y)$, then $p(x, x \sqcap y) = \sum_{n \in x \sqcap y = x \sqcap (x \sqcap y)} 2^{-n} = \sum_{n \in x \sqcap y} 2^{-n} = p(x,y)$.

It follows that $q(x, x \sqcap y) = p(x, x \sqcap y) - p(x,x) = p(x,y) - p(x,x) = q(x,y)$ for any $x, y \in P\omega$. Thus q is invariant , hence $(P\omega, \leq_p)$ is an invariant partial metric meet lattice. ∎

Example 4.2. Let X^∞ be the set of finite and infinite sequences over a non-empty set X, with the prefix ordering $(a_0, a1, \ldots, a_n) \leq (b_0, b_1, \ldots, b_m)$ if $n \leq m$ and $a_i = b_i$ for $i = 0, \ldots, n$. Denote the length of a sequence $x \in X^\infty$ by $l(x)$ with $l(\emptyset) = 0$, which is the index of the last term of x whose value is defined.

Then the function $p : X^\infty \times X^\infty \longrightarrow \mathbb{R}^+$ defined for any $x, y \in X^\infty$ by

$$p(x, y) = 2^{-\sup\{i \in \mathbb{N}: i \leq \min\{l(x), l(y)\}, \forall \ \ 0 \leq j < i, \ \ x_j = y_j\}}$$

is a partial metric on X^∞, called the *Baire partial metric*.

The value of the supremum is the first instance where the sequences differ (taking care if one sequence is shorter than the other).

Proof. First, let us denote for any $x, y \in X^\infty$,
$l(x, y) = \sup\{i \in \mathbb{N} : i \leq \min\{l(x), l(y)\}, \forall \ \ 0 \leq j < i, \ \ x_j = y_j\}$.
P1. We have that for any $x \in X^\infty, p(x, x) = 2^{-l(x)}$, and since for any $x, y \in X^\infty$, $l(x) \geq l(x, y)$, it then follows that $p(x, x) \leq p(x, y)$ for any $x, y \in X^\infty$.

P2. Suppose that for any $x, y \in X^\infty, p(x, x) = p(x, y) = p(y, y)$, then $l(x) = l(x, y) = l(y)$ and hence for any $j \leq l(x) = l(y), x_j = y_j$, thus $x = y$.

P3. For any $x, y \in X^\infty, l(x, y) = l(y, x)$, thus for any $x, y \in X^\infty$, $p(x, y) = p(y, x)$.

P4. Let $x, y, z \in X^\infty$. We have that

$$2^{-l(x,y)} + 2^{-l(y,z)} = \frac{1}{2^{l(x,y)}} + \frac{1}{2^{l(y,z)}},$$

and

$$2^{-l(x,z)} + 2^{-l(y,y)} = \frac{1}{2^{l(x,z)}} + \frac{1}{2^{l(y,y)}}.$$

Consider two cases:

- Suppose that $l(x, y) \leq l(x, z)$, then $2^{l(x,y)} \leq 2^{l(x,z)}$ and then $\frac{1}{2^{l(x,y)}} \geq \frac{1}{2^{l(x,z)}}$.

Since $\frac{1}{2^{l(y,z}} \geq \frac{1}{2^{l(y,y)}}$, then $2^{-l(x,y)} + 2^{-l(y,z)} \geq 2^{-l(x,z)} + 2^{-l(y,y)}$ which implies $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

- Suppose that $l(x, y) \geq l(x, z)$, then for any $0 \leq j \leq l(x, z), x_j = z_j = y_j$,

hence $l(x, z) \geq l(z, y)$ and then we have that $2^{l(x,z)} \geq 2^{l(z,y)}$, and that $\frac{1}{2^{l(z,y)}} \geq \frac{1}{2^{l(x,z)}}$.

Since $\frac{1}{2^{l(x,y}} \geq \frac{1}{2^{l(y,y)}}$, then $2^{-l(x,y)} + 2^{-l(y,z)} \geq 2^{-l(x,z)} + 2^{-l(y,y)}$ which implies that $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

Hence p is a partial metric. ∎

Proposition 4.0.10. *X^∞ with its partial metric is complete.*

Proof. Suppose that p is the partial metric of X^∞. Let us prove that (X^∞, p) is complete directly, that is every Cauchy sequence has a proper limit.

Let (x_n) be a Cauchy sequence in (X^∞, p), for any $n, m \geq 1$, there exists $t_1 \in \mathbb{N} \cup \{\infty\}$ such that $p(x_n, x_m) = 2^{-t_1}$. Then there exists $t \in \mathbb{N} \cup \{\infty\}$ so that $\lim_{n,m\to+\infty} p(x_n, x_m) = 2^{-t}$. For any $0 \leq k \leq t$, there exists $N_k \geq 1$ such that for any $n, m \geq N_k$, $l(x_n, x_m) > k$, obtained by using the fact that

$$\begin{aligned} l(x_n, x_m) > k &\Longleftrightarrow l(x_n), l(x_m) > k \text{ and } \forall j \leq k, x_{n,j} = x_{m,j} \\ &\Longleftrightarrow p(x_n, x_m) < 2^{-k}. \end{aligned}$$

Consider $y = (y_0, y_1, \dots) \in X^\infty$ where $y_k = x_{N_k}$ for all $0 \leq k < t$. Thus $l(y) = t$ and $p(y, y) = 2^{-t} = \lim_{n,m\to+\infty} p(x_n, x_m)$.

- Suppose that $t < +\infty$, then for any $n \geq \max\{N_i : 0 \leq i < t\}$ and $0 \leq i < t$, $l(x_n, x_{N_i}) > i$ (so $l(x_n) \geq t$) thus $x_{n,i} = x_{N_i,i} = y_i$, hence $l(x_n, y) = t$ and $p(x_n, y) = 2^{-t}$.

- Suppose now that $t = +\infty$. Given $\varepsilon > 0$, there exists $k \geq 0$ such that $0 < 2^{-k} < \varepsilon$. For any $n \geq \max\{N_i : 0 \leq i < k\}$ and for $0 \leq i < k$ ($k < t = +\infty$), $l(x_n, x_{N_i}) > i$ and $x_{n,i} = x_{N_i,i} = y_i$. This implies that for any $n \geq \max\{N_i : 0 \geq i < k\}$, $l(x_n, y) \geq k$, hence $p(x_n, y) \leq 2^{-k} < \varepsilon$.

In each case, we have that $\lim_{n\to+\infty} p(x_n, y) = 2^{-t} = \lim_{n,m\to+\infty} p(x_n, x_m) = p(y, y)$, in other words $x_n \to y$ (properly). Thus the sequence (x_n) converges to y properly in (X^∞, p), and we conclude that (X^∞, p) is complete. ∎

Proposition 4.0.11. (X^∞, p) *is an invariant partial metric meet semilattice.*

Proof. Let $(X^\infty, \leq_p)$ the partial order induced by (X^∞, p) defined for any $x, y \in X^\infty$ by $x \leq_p y \iff l(x) = l(x, y) \iff$ for any $0 \leq j < l(x), x_j = y_j$. It is a meet semilattice since for any $x, y \in X^\infty$, $x \sqcap y$ exists and is given by the longest common sequence with length $l(x \sqcap y) = l(x, y)$, and for which, for any $0 \leq j < l(x \sqcap y), x_j = y_j$.

It is also clear that it is invariant since for any $x, y \in X^\infty$,the longest common sequence of x and y is the same as the longest common sequence of x and $x \sqcap y$, that is $p(x, x \sqcap y) = p(x, y)$.

Thus if q is the induced quasi-metric defined for any $x, y \in X^\infty$ by $q(x, y) = p(x, y) - p(x, x)$, then $q(x, x \sqcap y) = p(x, x \sqcap y) - p(x, x) = p(x, y) - p(x, x) = q(x, y)$. Thus q is invariant, hence (X^∞, p) is an invariant partial metric meet semilattice. ∎

Conclusion

Partial metrics are more flexible than metrics, they generate partial orders and their topological properties are more general than the ones for metrics, argued by the fact that the self distance of each point need not be zero. They are enormously useful in partially defined informations for the study of domains and semantics in computer science.

We succeeded in developing the mathematical concepts of partial metrics, which are equivalent to weightable quasi-metrics in the sense that a partial metric can be interpreted as a weightable quasi-metric, and conversely a weighted quasi-metric can be considered as a partial metric.

The partial metric spaces were assigned every property from their induced metric spaces, allowing us to construct the completion of an incomplete partial metric space by first completing their induced metric space, using the set of all Cauchy sequences in the space of interest. The demonstration was similar to the one for completing metric spaces, involving notions of isometry and topological properties.

We then extended Banach's fixed point theorem to a complete partial metric space, using either a contractive mapping or a monotone non-decreasing function to demonstrate the existence and uniqueness of fixed points. Banach's fixed point theorem occurs in many mathematical fields especially in computer science for comparing semantic models.

We also presented the correspondence between partial metrics and co-weightable quasi-metrics, in a way similar to the equivalence between partial metrics and weightable quasi-metrics. We then introduced the notion of semilattices which involved partially ordered sets, allowing us to prove the bijection between fading strictly increasing semivaluations and invariant partial metric semilattices by using co-weightable quasi-metric spaces.

We finally illustrated the applications of partial metric spaces in computer science by using two concrete examples, known as domains.

Appendix A. Background

Partial metrics

Example A.1. Consider the usual metric d on $\mathbb{R}$ defined by $d(x,y) = |x-y|$ for any $x, y \in \mathbb{R}$, the function $p^f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ given by $p^f(x,y) = \frac{1}{2}d(x,y)$ for any $x, y \in \mathbb{R}$ is a partial metric which induces a flat order.

Proof. P1. For any $x, y \in \mathbb{R}, \quad p^f(x,y) \geq 0 = p(x,x)$.
P2. Suppose that for any $x, y \in \mathbb{R}, p^f(x,x) = p^f(x,y) = p^f(y,y)$, then $p(x,y) = 0$, $d(x,y) = 0$ hence $x = y$.
P3. Since $d(x,y) = d(y,x)$ for any $x, y \in \mathbb{R}$, it follows that $p(x,y) = p(y,x)$ for any $x, y \in \mathbb{R}$.
P4. For any $x, y, z \in \mathbb{R}$, one has

$$\begin{aligned} p^f(x,z) =& \frac{1}{2}|x-z| \\ \leq& (|x-y+y-z|) \\ \leq& \frac{1}{2}|x-y| + \frac{1}{2}|y-z| - \frac{1}{2}|y-y| \\ \leq& p^f(x,y) + p^f(y,z) - p^f(y,y). \end{aligned}$$

The induced order $\sqsubseteq$ defined by $x \sqsubseteq y \iff p^f(x,y) = p^f(x,x)$ for any $x, y \in \mathbb{R}$ is flat since $p(x,x) = p(x,y) \iff x = y$ as $p(x,x) = 0$. Hence $x \sqsubseteq y \iff x = y$ for any $x, y \in \mathbb{R}$, thus the order $\sqsubseteq$ is flat. ■

General topology

Definition A.1. A *topological basis* on a non-empty set X is a collection B of subsets of X satisfying

B1. $\bigcup_{U \in B} U = X$

B2. $U, V \in B \ \text{ and } \ x \in U \cap V \Rightarrow \exists W \in B \ \ x \in W \subseteq U \cap V$

The collection B generates a *topological space* $(X, \mathcal{T}(B))$ where $\mathcal{T}(B)$ is the *topology* induced by B given by $\mathcal{T}(B) = \{U \subseteq X : \forall x \in U \ \ \exists V \in B \ \ x \in V \subseteq U\}$. The elements of B are called the *basic open sets* of the space $(X, \mathcal{T}(B))$.

Metrics

Proposition A.1. *Let (X, d) be a metric space, B_d the topological basis induced by d and $(X, \mathcal{T}[\mathcal{B}])$ the corresponding topological space. Then, $(X, \mathcal{T}[\mathcal{B}])$ is a Hausdorff space. ss*

Proof. Suppose that $x_1 \neq x_2$, let $r = d(x_1, x_2) > 0$ and consider $U_j = B_{r/2}(x_j), \quad (j = 1, 2)$. Then $U_j, \quad (j = 1, 2)$ is open and $U_1 \cap U_2 = \emptyset$, one has $x_1 \in U_1, x_2 \in U_2$. Thus X is a Hausdorff space. ■

Proposition A.2. *Any isometric image of a complete metric space is complete.*

Proof. Let $j : X \longrightarrow Y$ denote any surjective isometric map from a complete metric space (X, d) into a metric space (Y, e).

Consider a Cauchy sequence (x_n) in Y. Since j is surjective, there exists a sequence (y_n) in X satisfying $x_n = j(y_n)$, and since j is an isometry, one has

$$e(x_m, x_n) = e[j(y_m), j(y_n)] = d(y_m, y_n)$$

for all $m, n \in \mathbb{N}$. Since (x_n) is a Cauchy sequence, then (y_n) is a Cauchy sequence in X. Thus (y_n) converges to a point $p \in X$. Let $q = j(p) \in Y$. Since j is an isometry, one has

$$e(x_n, q) = e[j(y_n), j(p)] = d(y_n, p)$$

for any $n \in \mathbb{N}$. Since $y_n \to p$, then $x_n \to q$. Hence Y is complete. ■

Proposition A.3. *A function $f : E \longrightarrow F$ of the metric spaces $(E, d_1), (F, d_2)$ is continuous at $x_0 \in E$ if and only if for every set V from a basis of neighbourhoods of $f(x_0)$, the pre-image $f^{-1}(V)$ is neighbourhood of x_0. It is continuous if and only if the pre-images of open (closed) sets in $(F, \mathcal{T}_2)$ are open (closed) in $(E, \mathcal{T}_1)$.*

Definition A.2. Let (X, d_1) and (Y, d_2) be metric spaces. A map $f : (X, d_1) \longrightarrow (Y, d_2)$ is said to be *uniformly continuous* if, for each given $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_2[f(x), f(y)] < \varepsilon$ for all $x, y \in X$ satisfying $d_1(x, y) < \delta$. An isometry is an example of uniformly continuous map.

Partial orders

Definition A.3. Let $(P, \leq_1)$ and $(Q, \leq_2)$ be partial orders. A function $f : P \longrightarrow Q$ is *increasing (decreasing)* if for any $x, y \in P$, $x \leq_1 y$ implies $f(x) \leq_2 f(y)$ $(f(y) \leq_2 f(x))$.

Definition A.4. A partially ordered set $(X, \leq)$ is flat if, for every $x, y \in X$, $x \leq y$ if, and only if, $x = y$.

Lemma A.4. *Suppose X is a topological space with specialisation order $\sqsubseteq$, then:*

1. *If X is a T_0 space, then $\sqsubseteq$ is a partial order.*
2. *If X is a T_2 space, then $\sqsubseteq$ is a flat partial order.*

Proposition A.5. *Let (X, d) be a quasi-metric space, then the relation $\leq$ defined by setting $x \leq y \Longleftrightarrow d(x, y) = 0$ for every $x, y \in X$ is a partial order.*

Conversely, let the relation $\leq$ be any partial order in a set X and define the mapping $d : X \times X \to [0, \infty)$ by setting

$$d(x, y) = \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

One can show that (X, d) is a quasi-metric space whose associated partial order is $\leq$.

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