Option Valuation using the Fourier Transform Method

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Abstract

Option pricing is one of the most important areas in financial mathematics and incorporates many different disciplines in mathematics. One such example is the use of the Fourier transform. Peter Carr and Dilip Madan in 1998 [CM98] developed it to compute the option price numerically by using the fast Fourier transform. In fact, making a change to the option price function to enforce integrability, we can calculate its Fourier transform knowing the characteristic function of the underlying asset. The Black-Scholes model can be used to determine the analytic form of the option value as a function of the strike. Therefore we can obtain the option price by giving an analytic expression of the Fourier transform of the modified value and get the price by Fourier inversion. The essay aimed at computing the option value numerically following this method and using the fast Fourier transform algorithm which is popular due to its fast calculation.

Contents

Introduction

Many different financial models can be used to find explicit formulas for the characteristic function of the underlying asset. The Black-Scholes formula for pricing an option is one of the easiest and fastest computations available. Many people have improved the Black-Scholes model in order to have a more realistic one. Such improvements include the Heston and Bates models (See [Hes93] and [Bat96]). The Black-Scholes model satisfies important market assumptions such as no arbitrage and is a widely used starting point in the study of financial markets in continuous time.

Fourier Analysis is a tool that has been used to compute the value of the option in a whole range of strikes due to its efficiency. Fourier Analysis has the advantage of permitting us to use spectrum of frequency components of a function and apply the Fourier inversion to have the function itself.

Our goal is to price the European call option by this Fourier analytic method. The essay is organised as follows:

In the first chapter, we will give an expression of the call value of the European option given the strike. For that, we will introduce some definitions, properties and theorems to understand the possible characteristics of the market. We are also going to give a notion of stochastic integral and martingales. Afterwards, we will develop the Black-Scholes model with its assumptions. We will state the famous Girsanov's theorem and its proof in order to show the existence of the martingale measure in the Black-Scholes market model which leads to the expression of the fair value.

The second chapter will be on the Fourier transform part. First, we will give a brief review of the Fourier transform. After that, we will see how to damp the option value function so that we can get a square integrable function and therefore we can apply the Fourier transform and its inverse on it. We also develop in this chapter the truncation error of the integral from the Fourier inversion which gives the option value function.

Lastly, in Chapter 3, which is the numerical part, we will talk briefly about the FFT-algorithm and develop the application of it to the option price function we obtained in the previous chapter. To test if the result is accurate, we will use the exact solution from the Black-Scholes formula for the comparison. And to test the speed of the algorithm, a Monte Carlo implementation will be used.

1. Pricing the European Call Option Using the Black-Scholes Model

Throughout this work, we consider a market with 2 assets S_0 and S_1 where S_0 is considered as the riskless bond with interest rate $r > 0$ and S_1 the risky asset. Our goal in this chapter is to be able to price a European call option for our market at maturity time T. Recall that a European call option gives the owner the right (but not the obligation) to buy an asset at the exercise time T. The payoff of European call option is given by $P_t^C = \max\{(S_t^1 - K, 0)\}$ where K is the strike price of the underlying asset at time T called the maturity time.
In order to price the option, we will give some ideas of stochastic calculus and martingales, define the Black-Scholes model and use the martingale measure to define and show the completeness of the market so that we can price the option in a fair manner.

1.1 Martingales and the Stochastic Calculus

1.1.1 Definitions and Properties

Definition 1.1. A stochastic process $X = \{X_t, t > 0\}$ is a family of random variables X_t which take values in $\mathbb{R}$ and are defined on the same probability space $(\Omega, \mathcal{F}, P)$.

Definition 1.2. We can define a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$ as a family of sets of information $\subset \mathcal{F}$ available up to time t and it has the property that $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$ and $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for $t \geq 0$.

Definition 1.3. A process X is adapted if X_t is $\mathcal{F}_t$-measurable for all $t > 0$. In other words if X_t can be known only up to time t.

Definition 1.4. A Brownian Motion is a continuous stochastic process W on $\mathbb{R}_+$ satisfying the following properties:

1. $W_0 = 0$.
2. $W_t - W_s$ for $t \geq s$ is independent of $\mathcal{F}_s$.
3. $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$, where $0 \leq s < t$.

1.1.2 Martingales and the Stochastic Calculus

We still consider the probability space $(\Omega, \mathcal{F}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\in\mathbb{R}_+}$.
Before coming to the definition of martingales, let us give some properties of random variables

using their conditional expectation.

Properties 1.5. *If $\mathcal{G} \subseteq \mathcal{F}$, we have the following properties:*

1. $E[E[X|\mathcal{G}]] = E[X]$.
2. $E[X|\mathcal{G}] = E[X]$ *if X is independent of $\mathcal{G}$.*
3. $E[X|\mathcal{G}] = X$ *if X is measurable with respect to $\mathcal{G}$.*

Definition 1.6. A moment generating function of a positive random variable X is a function g_X defined for $\alpha \in \mathbb{R}$ by

$$g_X(\alpha) = E[\exp(\alpha X)].$$

Two random variables $X \geq 0$ and $Y \geq 0$ are independent if $g_{X+Y}(\alpha) = g_X(\alpha)g_Y(\alpha)$.

Definition 1.7. A continuous process $\{S_t\}_{t\in\mathbb{R}_+}$ is a martingale if:

1. $S_t \in \mathcal{F}_t$ i.e S_t is adapted with respect to $\mathcal{F}_t$.
2. $E[|S_t|] < \infty$ (S_t is integrable).
3. $E[S_t|\mathcal{F}_s] = S_s$, for all $s \leq t$.

The following theorem is very useful since it gives the martingale property of stochastic integrals. The proof of the theorem can be found in [WDH93].

Theorem 1.8. *For any square integrable process G , the process X, defined by*

$$X_t = \int_0^t G_s dW(s)$$

is an $\mathcal{F}_t$-martingale.

Corollary 1.9. *With the same assumptions as in the previous theorem, a stochastic process X, satisfying a stochastic differential equation, is a martingale if and only if the stochastic differential has the form*

$$dX_t = G_t dW_t.$$

The proof is in [WDH93].

To solve a differential equation, we need the following theorem:

Theorem 1.10 (Ito's Formula). *Let f: $\mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \in C^2(\mathbb{R})$ and let X be a stochastic continuous process satisfying*

$$dX_t = \mu(t)dt + \sigma(t)dW_t,$$

where μ and σ are deterministic processes. Then the process Y such that $Y_t = f(t, X_t)$ satisfies the stochastic differential equation

$$dY_t = D_t f(\cdot, X_t)dt + D_x f(\cdot, X_t)dX_t + \frac{1}{2}D_{xx} f(\cdot, X_t)(dX_t)^2$$

with the following properties:

$$dW_t dt = 0, \quad dtdt = 0, \quad dW_t dW_t = dt.$$

See [Oks03] for the proof.

We are now able to compute the price of the option using the Black-Scholes model.

1.2 The Black-Scholes Model and the Risk Neutral Measure

The Black-Scholes model was proposed by the economist P. Samuelson in 1965 [Sam65]. In the context of option pricing, it is the standard and most used model in continuous time. The Black-Scholes model is the limit of the binomial model in discrete time. Before developing the model we will give some definitions in order to understand the characteristic of the market.

Definition 1.11. A trading strategy is self-financing if investors in the model only make trades that they can afford.

Definition 1.12. An arbitrage opportunity is the possibility to make profit in the market without investing any capital and without any risk. The principle of no-arbitrage consits of not allowing mathematical models in finance to have arbitrage opportunities.

Theorem 1.13 (Fundamental Theorem of Asset Pricing)**.** *For a financial market modelled on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$, the following are equivalent:*

1. *The market satisfies the no-arbitrage principle.*
2. *There exists an equivalent martingale measure.*

The proof can be found in [DS05].

Definition 1.14. A derivative security D with maturity time T can be replicated if there exists a self-financing strategy h such that the value V^h of the portfolio at time T is equal to D i.e $V^h(T) = D$

Definition 1.15. A market is complete if every derivative security of it is replicable.

Theorem 1.16 (Second Fundamental Theorem of Asset Pricing)**.** *With the assumption that there are no arbitrage opportunities, the market is complete if and only if there is a unique equivalent martingale measure Q. This theorem can be seen in [Mar07].*

The Black-Scholes model which we are going to use here is the 1-dimensional Black-Scholes model.

1.2.1 The Model and its Assumptions

Considering our framework in the beginning of the chapter and assuming that S^1 is a stochastic process, the Black-Scholes model for our market is the following:

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ S_0^0 = 1. \end{cases}$$

and:

$$\begin{cases} dS_t^1 = \mu S_t^1 dt + \sigma S_t^1 dW_t \\ S_0^1 > 1. \end{cases}$$

This model allows us to determine the fair price at time zero for a European call or put in terms of the initial risky asset price S_0^1, the strike price K, the asset volatility σ, the risk-free rate r and the maturity time T. While we are deriving the fair price of the option, we will see that it does not depend on the drift μ which determines the expected growth of the risky asset.

The Black-Scholes analysis of the market has the following assumptions that we can find in [WDH93].

- The price follows the lognormal random walk. That is to say the natural logarithm of the stock prices are normally distributed.
- In the Black-Scholes analysis, the risk-free rate and the asset volatility σ are deterministic. For us, to make it more simpler we are going to consider them constants.
- We assume that hedging a portfolio does not involve transaction costs
- The underlying asset does not pay any dividend up to the maturity time of the option.
- There are no arbitrage opportunities.
- We can buy and sell underlying assets as much as we want.

As the Black-Scholes model is the equivalence of the discrete binomial model, the following theorem holds:

Theorem 1.17. *Suppose that D is a replicable derivative security with replicating strategy h. The fair price at any time $t \in [0, T]$ is equal to the value at time t of the portfolio that finances D. See [DB02] for the proof.*

Theorem 1.18 (Girsanov's Theorem). *1. Let Z be the process such that:*

$$Z_t = \exp\left\{\int_0^t H_s dW_s - \frac{1}{2}\int_0^t H_s^2 ds\right\}, \tag{1.1}$$

where H is a process satisfying, for each $t \geq 0$,

$$E(\exp\{\int_0^t \frac{H_s^2}{2} ds\}) < \infty \tag{1.2}$$

Then Z is the unique solution to the stochastic differential equation:

$$dZ_t = Z_t H_t dW_t \qquad Z_0 = 1 \tag{1.3}$$

which satisfies $E(Z_t) = 1$ for all $t \in [0, T]$. The process Z is a martingale.

2. *Let Q be the equivalent probability measure defined on $(\Omega, \mathcal{F}_t, P)$ by $\frac{dQ}{dP} = Z_t$. Under Q the process $\widetilde{W}_t = W_t - \int_0^t H_s ds$ is a Brownian motion under the measure Q. See [DB02].*

Since later we only need the case where H is a deterministic process, we restrict the proof of the theorem to this special case.

Proof. To solve the stochastic differential equation (1.3), we write

$$\frac{dZ_t}{Z_t} = H_t dW_t, \tag{1.4}$$

and applying the Ito's formula to the function $f(t, Z_t) = \ln Z_t$ with the assumptions stated in Theorem 1.10,

$$d(\ln(Z_t)) = \frac{dZ_t}{Z_t} - \frac{1}{2} H_t^2 dt.$$

By integrating under the condition $Z_0 = 1$, we have:

$$\begin{aligned} \ln(Z_t) &= \int_0^t \frac{dZ_s}{Z_s} - \int_0^t \frac{1}{2} H_s^2 ds \\ &= \int_0^t H_s dW_s - \int_0^t \frac{1}{2} H_s^2 ds, \end{aligned}$$

and we therefore obtain the Equation (1.1). The uniqueness is given by the initial condition $Z_0 = 1$. Since H_t is not random and W_t is a Brownian motion, then by the properties of the normal distribution, the random variable $\int_0^t H_s dW_s$ is normally distributed with mean 0 and variance $\int_0^t H_s^2 ds$. Using also the fact that for a random variable X normally distributed with mean 0 and variance σ^2, $E[\mathrm{e}^X] = \mathrm{e}^{\sigma^2/2}$, we get

$$\begin{aligned} E[Z_t] &= E\left[\mathrm{e}^{\int_0^t H_s dW_s} \mathrm{e}^{-\frac{1}{2}\int_0^t H_s^2 ds}\right] \\ &= \mathrm{e}^{-\frac{1}{2}\int_0^t H_s^2 ds} E\left[\mathrm{e}^{\int_0^t H_s dW_s}\right] \\ &= \mathrm{e}^{-\frac{1}{2}\int_0^t H_s^2 ds} \mathrm{e}^{\frac{1}{2}\int_0^t H_s^2 ds} \\ &= 1. \end{aligned}$$

For any $s < t$, we have also that the random variable $\int_s^t H_u dW_u$ is normally distributed and independent of $\mathcal{F}_s$. Therefore

$$\begin{aligned} E\left(\exp\left\{\int_0^t H_u dW_u\right\}\middle|\mathcal{F}_s\right) &= \exp\left\{\int_0^s H_u dW_u\right\}E\left(\exp\left\{\int_s^t H_u dW_u\right\}\middle|\mathcal{F}_s\right) \\ &= \exp\left\{\int_0^s H_u dW_u\right\}E\left(\exp\left\{\int_s^t H_u dW_u\right\}\right) \\ &= \exp\left\{\int_0^s H_u dW_u\right\}\exp\left\{\int_s^t \frac{H_u^2}{2} du\right\}. \end{aligned}$$

Hence

$$\begin{aligned} E[Z_t|\mathcal{F}_s] &= \exp\left\{\int_0^s H_u dW_u\right\}\exp\left\{\int_s^t \frac{H_u^2}{2} d_u\right\}\exp\left\{-\int_0^t \frac{H_u^2}{2} du\right\} \\ &= \exp\left\{\int_0^s H_u dW_u\right\}\exp\left\{-\int_0^s \frac{H_u^2}{2} du\right\} \\ &= Z_s. \end{aligned}$$

Hence, the first part of the theorem is proven. Now, let us prove that the process $\widetilde{W}$ defined by $\widetilde{W}_t = W_t - \int_0^t H_s ds$ is a Q-Brownian Motion. Since the process W is a Brownian motion, the process $\widetilde{W}$ satisfies 1 of Definition 1.4. To show that it satisfies 2 and 3 i.e $\widetilde{W}_t - \widetilde{W}_s$ for $t \geq s$ is independent of $\mathcal{F}_s$ and is normally distributed with mean 0 and variance $t - s$, we use the property of random variables in Definition 1.6. We have also the fact that for a normally distributed random variable X with mean 0 and variance σ^2, $E[\exp X] = \exp\left\{\frac{\sigma^2}{2}\right\}$. Then, for the increments

$$\widetilde{W}_{t_1},\ \ \widetilde{W}_{t_2} - \widetilde{W}_{t_1}, \cdots,\ \ \widetilde{W}_{t_n} - \widetilde{W}_{t_{n-1}}$$

to be both independent and normally distributed with expectation 0 and variances $t_1,\ t_2 - t_1, \cdots,\ t_n - t_{n-1}$ we have to show the following equality:

$$E_Q\left[\exp\left\{\sum_{k=1}^n \alpha_k\left(\widetilde{W}_{t_k} - \widetilde{W}_{t_{k-1}}\right)\right\}\right] = \prod_{k=1}^n \exp\left\{\frac{\alpha_k^2}{2}(t_k - t_{k-1})\right\} \tag{1.5}$$

Using the fact that

$$\widetilde{W}_{t_k} - \widetilde{W}_{t_{k-1}} \stackrel{d}{=} \widetilde{W}_{t_k - t_{k-1}}$$

in distribution, we have

$$E_Q\left[\exp\left\{\sum_{k=1}^n \alpha_k\left(\widetilde{W}_{t_k} - \widetilde{W}_{t_{k-1}}\right)\right\}\right] = E_Q\left[\exp\left\{\sum_{k=1}^n \alpha_k \widetilde{W}_{t_k - t_{k-1}}\right\}\right]. \tag{1.6}$$

Then

$$\begin{aligned}
E_Q\left[\exp\left\{\sum_{k=1}^{n}\alpha_k\widetilde{W}_{t_k-t_{k-1}}\right\}\right] &= E_Q\left[\exp\left\{\sum_{k=0}^{n}\alpha_k\left(W_{t_k-t_{k-1}}-\int_0^{t_k-t_{k-1}}H_s ds\right)\right\}\right]\\
&= E_Q\left[\prod_{k=1}^{n}\exp(\alpha_k W_{t_k-t_{k-1}})\prod_{k=1}^{n}\exp\left(-\alpha_k\int_0^{t_k-t_{k-1}}H_s ds\right)\right]\\
&= \prod_{k=1}^{n}\exp\left(-\alpha_k\int_0^{t_k-t_{k-1}}H_s ds\right)\prod_{k=1}^{n}E_P\Bigg[\exp(\alpha_k W_{t_k-t_{k-1}})\\
&\qquad\times\exp\left\{\int_0^{t_k-t_{k-1}}H_s dW_s-\int_0^{t_k-t_{k-1}}\frac{H_s^2}{2}ds\right\}\Bigg]\\
&= \prod_{k=1}^{n}E_P\Bigg[\exp\Bigg\{\int_0^{t_k-t_{k-1}}(\alpha_k+H_s)dW_s\\
&\qquad-\int_0^{t_k-t_{k-1}}(2\alpha_k H_s+H_s^2)\frac{ds}{2}\Bigg\}\Bigg]\\
&= \prod_{k=1}^{n}\exp\left\{\alpha_k^2(t_k-t_{k-1})/2\right\}\prod_{k=1}^{n}E_P\Bigg[\exp\Bigg\{\int_0^{t_k-t_{k-1}}(\alpha_k+H_s)dW_s\\
&\qquad-\int_0^{t_k-t_{k-1}}(\alpha_k+H_s)^2\frac{ds}{2}\Bigg\}\Bigg].
\end{aligned} \tag{1.7}$$

From the first part of the theorem, we have just proven, we have that

$$E_P\left[\exp\left\{\int_0^{t_k-t_{k-1}}(\alpha_k+H_s)dW_s-\int_0^{t_k-t_{k-1}}(\alpha_k+H_s)^2\frac{ds}{2}\right\}\right]=1 \tag{1.8}$$

Then Equation (1.6) is equal to

$$\prod_{k=1}^{n}\exp\left\{\frac{\alpha_k^2(t_k-t_{k-1})}{2}\right\}$$

and we have proven Equation (1.5). □

We are now be able to proof the following theorem which allows us to determine the price of the option.

Theorem 1.19. *In the 1-dimensional Black-Scholes model, there exists a probability measure Q that is equivalent to P and such that the discounted prices are martingales under Q.*

Proof. Proving the theorem consists of finding Q such that the discounted prices are martingales under Q.

$\bar{S}_t^1 = \mathsf{e}^{-rt} S_t^1$ is the discounted price of the risky asset. Then

$$\begin{aligned} d\bar{S}_t^1 &= d(\mathsf{e}^{-rt} S_t^1) \\ &= -r\mathsf{e}^{-rt} S_t^1 dt + \mathsf{e}^{-rt} dS_t^1 \\ &= -r\mathsf{e}^{-rt} S_t^1 dt + \mathsf{e}^{-rt}(\mu S_t^1 dt + \sigma S_t^1 dW_t) \\ &= \mathsf{e}^{-rt} S_t^1 [(\mu - r)dt + \sigma dW_t] \\ &= \bar{S}_t^1 [(\mu - r)dt + \sigma dW_t]. \end{aligned}$$

Hence

$$\int_0^t \frac{d\bar{S}_s^1}{\bar{S}_s^1} = (\mu - r)t + \sigma W_t.$$

To find $\int_0^t \frac{d\bar{S}_s^1}{\bar{S}_s^1}$, we apply the Ito's formula to the function $f(x,t) = \ln x$, like we did in the proof of Girsanov's theorem and we obtain:

$$\begin{aligned} \ln \frac{\bar{S}_t^1}{\bar{S}_0^1} &= \int_0^t \frac{d\bar{S}_s^1}{\bar{S}_s^1} - \int_0^t \frac{1}{2}\sigma^2 ds \\ &= (\mu - r)t + \sigma W_t - \frac{1}{2}\sigma^2 t. \end{aligned}$$

Then

$$\begin{aligned} \bar{S}_t^1 &= \bar{S}_0^1 \exp\left((\mu - r)t + \sigma W_t - \frac{1}{2}\sigma^2 t \right) \\ &= \bar{S}_0^1 \exp\left(\sigma \left(\frac{(\mu - r)}{\sigma} t + W_t \right) - \frac{1}{2}\sigma^2 t \right). \end{aligned}$$

If we let Z be the process satisfying

$$dZ_t = \frac{-(\mu - r)}{\sigma} Z_t dW_t \quad \text{and} \quad Z_0 = 1,$$

then the second part of the Girsanov's Theorem shows that under the probability Q defined by $\frac{dQ}{dP} = Z_T$ on $\mathcal{F}_T$

$$\begin{aligned} \widetilde{W}_t &= W_t + \int_0^t \frac{\mu - r}{\sigma} ds \\ &= W_t + \frac{\mu - r}{\sigma} t \end{aligned}$$

is a Brownian motion under the probability Q. Since

$$\begin{aligned} \bar{S}_t^1 &= \bar{S}_0^1 \exp\left\{ \sigma \left(\frac{(\mu - r)}{\sigma} t + W_t \right) - \frac{1}{2}\sigma^2 t \right\} \\ &= \bar{S}_0^1 \exp\left\{ \int_0^t \sigma d\widetilde{W}_s - \frac{1}{2} \int_0^t \sigma^2 ds \right\} \end{aligned}$$

the discounted price $\bar{S}_t^1$ satisfies $d\bar{S}_t^1 = \bar{S}_t^1 \sigma d\widetilde{W}_t$ with $\bar{S}_0^1 = 1$, and $E(\bar{S}_t^1) = 1$ for all $t \in [0, T]$. By integration and Corollary 1.9 $\bar{S}_t^1$ is a martingale. Hence the risky asset S_t^1 satisfies

$$dS_t^1 = S_t^1(rdt + \sigma d\widetilde{W}_t) \tag{1.9}$$

and the discounted price $\bar{S}_t^1$ is a martingale under the measure Q. For the risk-free asset, it is obvious to see that under Q the discounted price is constant and equal to 1. Hence, $\bar{S}_t^0$ is also a Q-martingale. □

We can conclude then from Theorem 1.19 and Theorem 1.16 that the market modelled by the 1-dimensional Black-Scholes model is complete. (See Definition 1.15).

1.2.2 Pricing the Option using the Risk Neutral Measure

Now if we construct a portfolio such that its value V_t at time t satisfies

$$V_t = a_t S_t^0 + b_t S_t^1, \qquad \text{see} \quad \text{[DB02]}$$

under the probability Q from Theorem 1.19, we have:

$$\begin{aligned} dV_t &= a_t S_t^0 r dt + b_t S_t^1 (rdt + \sigma d\widetilde{W}_t) \\ &= V_t r dt + dM_t, \end{aligned}$$

where $dM_t = b_t S_t^1 \sigma d\widetilde{W}_t$ and M_t is a martingale after integration.

Applying the Ito's formula to $f(t, x) = \mathrm{e}^{-rt} x$ for $x = V_t$ we get:

$$d(\mathrm{e}^{-rt} V_t) = \mathrm{e}^{-rt} dM_t$$

Then:

$$\begin{aligned} \mathrm{e}^{-rt} V_t &= \int_0^t \mathrm{e}^{-rs} dM_s + V_0 \\ &= \int_0^t \mathrm{e}^{-rs} b_t S_s^1 \sigma d\widetilde{W}_s + V_0. \end{aligned}$$

Since $\widetilde{W}$ is a Brownian motion under Q, Theorem 1.8 tells us that $\mathrm{e}^{-rt} V_t$ is a martingale under Q. Then

$$\mathrm{e}^{-rt} V_t = E_Q(\mathrm{e}^{-rT} V_T | \mathcal{F}_t),$$

e^{-rt} is deterministic and V_T is nothing else but the payoff of the option since the market is replicable by Definition 1.14. Then, by Theorem 1.17

$$V_t = E_Q(\mathrm{e}^{-r(T-t)} V_T | \mathcal{F}_t)$$

is the fair value of the option at time t.

For a European call option, let k be the log of the strike price K and s the log of the price S of the underlying asset and let C_{T-t} be the fair value of the option at time t. Hence,

$$\begin{aligned} C_{T-t} &= E_Q(\mathsf{e}^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t) \\ &= E_Q(\mathsf{e}^{-r(T-t)}(\mathsf{e}^{s_T} - \mathsf{e}^k)^+|\mathcal{F}_t). \end{aligned}$$

If q_{T-t} is the risk neutral density of the log price of the underlying asset. Then at time t, the value of the call option is given by

$$C_{T-t}(k) = \int_k^\infty \mathsf{e}^{-r(T-t)}(\mathsf{e}^s - \mathsf{e}^k)q_{T-t}(s)ds$$

and the initial call value of the option, as we see in [CM98] is:

$$C_T(k) = \int_k^\infty \mathsf{e}^{-rT}(\mathsf{e}^s - \mathsf{e}^k)q_T(s)ds.$$

2. The Fourier Transform Method

2.1 Review of the Fourier Transform

Definition 2.1. The Fourier transform is a linear operator which transforms a function into a continuous range of its frequency components.

There are many conventions of giving a formula for the Fourier transform, for our work we are going to use the non unitary Fourier transform which is given for a function f by the following formula:

$$\mathcal{F}(\omega) = \int_{-\infty}^{+\infty} f(t)\mathrm{e}^{i\omega t}dt \qquad \forall\omega \in \mathbb{R}. \tag{2.1}$$

Then for a continuous function which is represented in its frequencies in time, we can have the function using the inverse Fourier transform

$$f(t) = \bar{\mathcal{F}}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathcal{F}(\omega)\mathrm{e}^{-i\omega t}d\omega. \tag{2.2}$$

We have not yet given the existence of the integral.

For Equation (2.2) to make sense, we need f to be integrable. But in this case, the inverse of the Fourier transform does not necessarily make sense. We have then to enlarge the domain where we compute the Fourier transform.

One can define the Fourier transform for square integrable functions. In fact, we have the following proposition:

Proposition 2.2. *The Fourier transform $\mathcal{F}$ is an isometry of $L^2(\mathbb{R})$ with inverse $\bar{\mathcal{F}}$ and it satisfies the Parseval equality which is:*

$$\int_{-\infty}^{+\infty} f_1(x)\bar{f}_2(x)dx = \int_{-\infty}^{+\infty} \mathcal{F}_1(\omega)\bar{\mathcal{F}}_2(\omega)d\omega.$$

Proof. See [Riv07] □

Then if f is a square integrable function, the Fourier transform of f and its inverse transform exist and make sense.

2.2 The Use of The Fourier Transform

From the first chapter, we have:

$$C_T(k) = \int_{-\infty}^{+\infty} \mathrm{e}^{-rT}(\mathrm{e}^s - \mathrm{e}^k)^+ q_T(s)ds.$$

Before Computing the Fourier Transform, let us give the expression of the characteristic function which we will need later on. The characteristic function of the density $q_T(s)$ of the log price is:

$$\Phi_T(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{ixs} q_T(s)ds.$$

Let $g(u) = u^{ix}$, then:

$$\begin{aligned}
\Phi_T(x) &= E[g(S_T^1)] \\
&= E\left[g\left(S_t^1 \exp\left\{(r - \frac{\sigma^2}{2})(T-t) + \sigma(\bar{W}_T - \bar{W}_t)\right\}\right)\right] \\
&= \int_{-\infty}^{+\infty} g(\mathrm{e}^Y) N(\log S_t^1 + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)))dY \\
&= E[\mathrm{e}^{ixY}] \text{ where } Y \sim N(\log S_t^1 + (r - \frac{\sigma^2}{2})(T-t), \sigma^2(T-t)) \\
&= \exp\left\{ix(\log S_t^1 + (r - \frac{\sigma^2}{2})(T-t)) - \frac{\sigma^2(T-t)x^2}{2}\right\},
\end{aligned}$$

which gives us at initial time $t = 0$:

$$\Phi_T(x) = \exp\left\{ix\left(\log S_0^1 + (r - \frac{\sigma^2}{2})T\right) - \frac{\sigma^2 T x^2}{2}\right\} \tag{2.3}$$

The square integrability of the call value function is required since we will compute the Fourier transform and the inverse of it. When the log strike price k tends to $-\infty$, we have:

$$\begin{aligned}
\lim_{k \to -\infty} C_T(k) &= \int_{-\infty}^{+\infty} \mathrm{e}^{-rT}\mathrm{e}^s q_T(s)ds \\
&= E_Q[\mathrm{e}^{-rT} S_T | \mathcal{F}_0] \\
&= S_0. \qquad (2.4)
\end{aligned}$$

which is non zero and therefore $C_T(k)$ is not integrable. In order to have an integrable function, hence square integrable, we are going to multiply $C_T(k)$ by $\mathrm{e}^{\alpha k}$ where $\alpha > 0$. The suitable value of α will be determined by the simulation.

Our modified call option value is then:

$$c_T(k) = \mathrm{e}^{\alpha k} C_T(k).$$

We denote the Fourier transform of the modified call value by $\mathcal{T}$ which is given by:

$$\mathcal{T}(u) = \int_{-\infty}^{-\infty} \mathsf{e}^{iuk} c_T(k) dk. \tag{2.5}$$

Let us give the analytical function of the integral (2.5) in terms of the characteristic function of the density function $q_T(s)$

$$\begin{aligned}
\mathcal{T}(u) &= \int_{-\infty}^{+\infty} \mathsf{e}^{iuk} \int_{-\infty}^{+\infty} \mathsf{e}^{\alpha k} e^{-rT} (\mathsf{e}^s - \mathsf{e}^k)^+ q_T(s) ds dk \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathsf{e}^{iuk} \mathsf{e}^{\alpha k} \mathsf{e}^{-rT} (\mathsf{e}^s - \mathsf{e}^k)^+ q_T(s) ds dk \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathsf{e}^{iuk} e^{\alpha k} \mathsf{e}^{-rT} (\mathsf{e}^s - \mathsf{e}^k)^+ q_T(s) dk ds \\
&= \int_{-\infty}^{+\infty} \mathsf{e}^{-rT} q_T(s) \int_{-\infty}^{s} \mathsf{e}^{iuk} \mathsf{e}^{\alpha k} (\mathsf{e}^s - \mathsf{e}^k) dk ds \\
&= \int_{-\infty}^{+\infty} \mathsf{e}^{-rT} q_T(s) \int_{-\infty}^{s} \mathsf{e}^{iuk+s+\alpha k} - \mathsf{e}^{iuk+(\alpha+1)k} dk ds \\
&= \int_{-\infty}^{+\infty} \mathsf{e}^{-rT} q_T(s) \left[\frac{\mathsf{e}^{iuk+s+\alpha k}}{iu+\alpha} - \frac{\mathsf{e}^{iuk+(\alpha+1)k}}{iu+\alpha+1} \right]_{-\infty}^{s} ds \\
&= \int_{-\infty}^{+\infty} \frac{\mathsf{e}^{(iu+1+\alpha)s} \mathsf{e}^{-rT} q_T(s)}{(iu+\alpha)(iu+\alpha+1)} ds \\
&= \frac{\mathsf{e}^{-rT} \Phi_T(u-(\alpha+1)i)}{(iu+\alpha)(iu+\alpha+1)}.
\end{aligned} \tag{2.6}$$

Now the computation of the inverse transform of $\mathcal{T}(u)$ gives

$$C_T(k) = \frac{\mathsf{e}^{-\alpha k}}{2\pi} \int_{-\infty}^{+\infty} \mathsf{e}^{-iuk} \mathcal{T}(u) du.$$

As $C_T(k)$ is real, we have $\overline{C_T(k)} = C_T(k)$. Changing the variable k into $-k$ in $\overline{C_0(k)}$, we have that the real part of $\mathcal{T}(u)$ is even and the imaginary part is odd. Therefore:

$$C_T(k) = \frac{\mathsf{e}^{-\alpha k}}{\pi} \int_{0}^{+\infty} \mathsf{e}^{-iuk} \mathcal{T}(u) du. \tag{2.7}$$

2.2.1 The Truncation Error

Now, because we will later calculate numerically $C_T(k)$, we have to think of how to truncate the infinite upper limit of the integration (2.7). Let us approximate this integral by:

$$\frac{\mathsf{e}^{-\alpha k}}{\pi} \int_{0}^{a} \mathsf{e}^{-iuk} \mathcal{T}(u) du.$$

We have that

$$\begin{aligned}|\int_0^{+\infty} \mathsf{e}^{-iuk}\mathcal{T}(u)du - \int_0^{a} \mathsf{e}^{-iuk}\mathcal{T}(u)du| &= |\int_a^{+\infty} \mathsf{e}^{-iuk}\mathcal{T}(u)du| \\ &\leq \int_a^{+\infty} |\mathcal{T}(u)|du.\end{aligned}$$

We also see that

$$\begin{aligned}|\phi_T(u-(\alpha+1)i)| &= |\int_{-\infty}^{+\infty} \mathsf{e}^{isu}\mathsf{e}^{(\alpha+1)s}q(s)ds| \\ &\leq \int_{-\infty}^{+\infty} |\mathsf{e}^{(\alpha+1)s}q(s)|ds \\ &= E[\mathsf{e}^{(\alpha+1)s}] \\ &= E[S_T^{\alpha+1}].\end{aligned}$$

Since $E[S_T^{\alpha+1}] < \infty$, we have from (2.6) for some constant A:

$$|\mathcal{T}(u)|^2 \leq \frac{E[S_T^{\alpha+1}]^2}{(\alpha^2+\alpha-u^2)^2+(2\alpha+1)^2u^2} \leq \frac{A}{u^4},$$

hence:

$$\int_a^{+\infty} |\mathcal{T}(u)|du < \frac{\sqrt{A}}{a},$$

and the truncation error is bounded by:

$$\frac{\mathsf{e}^{-\alpha k}}{\pi}\frac{\sqrt{A}}{a}.$$

If ε is the maximum error of the truncation then we must choose a such that:

$$a > \frac{\mathsf{e}^{-\alpha k}}{\pi}\frac{\sqrt{A}}{\varepsilon}.$$

3. Option Pricing Using the Fast Fourier Transform (FFT)

In order to compute the value of the option, the use of an efficient algorithm is required. In this chapter we are going to show why we can use the FFT and why the algorithm is efficient.

3.1 The Algorithm and its Complexity

The discrete Fourier transform is nothing else but the Fourier transform in discrete time and discrete frequencies. The FFT is an efficient algorithm for computing the discrete Fourier transform $X(k)$ of the function which we denote $x(j)$ where k and j are the frequencies and the time.

$$X(k) = \sum_{j=0}^{N-1} \mathrm{e}^{-i2\frac{\pi}{N}kj} x(j) \quad \text{for} \quad k = 0, \cdots, N. \tag{3.1}$$

A straightforward computation of this sum requires us N^2 complex multiplications and $N(N-1)$ complex additions. The FFT algorithm reduces the number of operations for the computation. For $N = 2^\gamma$, where γ is a positive integer, the Cooley-Tukey method for the FFT algorithm consists of factoring $N \times N$ matrix into γ matrices (each $N \times N$) such that they minimise the number of complex operation in the computation. Instead of N^2 complex multiplications, the Cooley-Tukey algorithm requires only $N \log_2 N$ operations. In order for us to understand the algorithm and to see the rapidity of it, let us show it for $N = 4$.

In (3.1), for $N = 2^2$ i.e $\gamma = 2$, we represent k and j as binary numbers.

$$\begin{aligned} k &= (k_1, k_0) \\ j &= (j_1, j_0) \qquad \text{where} \quad j_1, j_0, k_1, k_0 \in \{0, 1\}. \end{aligned}$$

Hence, we can write

$$\begin{aligned} k &= 2k_1 + k_0. \\ j &= 2j_1 + j_0. \end{aligned}$$

replacing k and j in (3.1) and denoting $W = \mathrm{e}^{-\frac{i2\pi}{N}}$ we get:

$$X(k_1, k_0) = \sum_{j_0=0}^{1} \sum_{j_1=0}^{1} x(j_1, j_0) W^{(2j_1+j_0)(2k_1+k_0)}.$$

Noticing that for $N = 4$, $W^{4j_1k_1} = 1$ we have:

$$X(k_1, k_0) = \sum_{j_0=0}^{1} \left[\sum_{j_1=0}^{1} x(j_1, j_0) W^{2k_0 j_1} \right] W^{(2k_1+k_0)j_0}.$$

Now, we rewrite

$$x_1(k_0, j_0) = \sum_{j_1=0}^{1} x(j_1, j_0) W^{2k_0 j_1},$$
$$x_2(k_1, k_0) = \sum_{j_0=0}^{1} x_1(k_0, j_0) W^{(2k_1+k_0)j_0},$$

and

$$X(k_1, k_0) = x_2(k_1, k_0).$$

These three equations represent the Cooley-Tukey method of the FFT algorithm. Citing x_1 and x_2 for k_0, k_1, j_0 and writing it in term of matrices help us to see easily that it requires only $8 = 4\log_2 4$ operations.

3.2 Application of the FFT to the Call Value Function

Now, let us return to our call option value. We want to have the call value in terms of a sum so that we can use the FFT. Using the Euler method to compute Equation (2.7) numerically, we have

$$C_T(k) \approx \frac{\mathrm{e}^{-\alpha k}}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-iu_j k} \mathcal{T}(u_j) h, \tag{3.2}$$

where $u_j = h(j-1)$ and h is the stepsize of the numerical integration. Then the upper limit of the integration is

$$a = Nh.$$

Let us consider a range of log strike price around the log initial price of the asset:

$$k_l = -\frac{1}{2}N\lambda + \lambda(l-1) \ \text{ for } \ l = 1, \cdots, N,$$

where $\lambda > 0$ is the distance between the log strike prices. Substituting these log strike prices into Equation (3.2) we obtain:

$$C_T(k_l) \approx \frac{\mathrm{e}^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-iu_j(-\frac{1}{2}N\lambda+\lambda(l-1))} \mathcal{T}(u_j) h. \tag{3.3}$$

$u_j = (j-1)h$ gives

$$C_T(k_l) \approx \frac{\mathrm{e}^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-i\lambda(j-1)(l-1)} \mathrm{e}^{i\frac{1}{2}(j-1)N\lambda h} \mathcal{T}(u_j) h. \tag{3.4}$$

In order to have the discrete Fourier transform in the form of Equation (3.1), we let

$$\lambda h = \frac{2\pi}{N}.$$

Hence

$$C_T(k_l) \approx \frac{\mathrm{e}^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-i\frac{2\pi}{N}(j-1)(l-1)} \mathrm{e}^{i(j-1)\pi} \mathcal{T}(u_j) h$$
$$= \frac{\mathrm{e}^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} (-1)^{j-1} \mathrm{e}^{-i\frac{2\pi}{N}(j-1)(l-1)} \mathcal{T}(u_j) h. \quad (3.5)$$

We can now apply the FFT to

$$x_j = (-1)^{j-1} \mathcal{T}(u_j) h \quad \text{for} \quad j = 1, \cdots, N. \quad (3.6)$$

One can choose strikes near the initial price S_0. In this case we should arrange the prices such that the initial price S_0 appears in the range of the strikes and we should choose a small value of λ in order to have many strikes around it. This implies a large value of h which gives a large grid for the integration. To obtain an accurate integration with large value of h, Carr and Madan in [CM98] propose to apply the Simpson's rule weightings into our Equation (3.5) which gives

$$C_T(k_l) \approx \frac{\mathrm{e}^{-\alpha k_l}}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-i\frac{2\pi}{N}(j-1)(l-1)} (-1)^{j-1} \mathcal{T}(u_j) \frac{h}{3} [3 + (-1)^j - \delta_{j-1}], \quad (3.7)$$

where δ_n is the Kronecker delta function given by

$$\delta_n = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then, Equation (3.6) becomes

$$x_j = (-1)^{j-1} \mathcal{T}(u_j) \frac{h}{3} [3 + (-1)^j - \delta_{j-1}] \quad \text{for} \quad j = 1, \cdots, N.$$

We can now apply the discrete FFT to x_j by choosing a small λ. For the first calculation, we use the following parameters: $S_0^1 = 1$ is the initial price of the underlying asset, and the interest rate $r = 0.02$. The value of α we use is $\alpha = 0.75$ (see [BDH05]). Our last parameter is the volatility which is chosen to be 0.7.

From Equation (2.4), the option price tends to the initial price S_0^1 of the underlying asset when the strike price K tends to zero and we can see this fact in the different plots.

The number N to compute the FFT increases the number of points but it does not affect the convergence of the FFT method to the analytical one very much. This is justified in Figure 3.1, Figure 3.2 and Figure 3.3 where we increase $N = 16$ to $N = 4096$.

Notice that this method works for any value of the initial price S_0, the result for $S_0 = 100$ is shown in Figure 3.4.

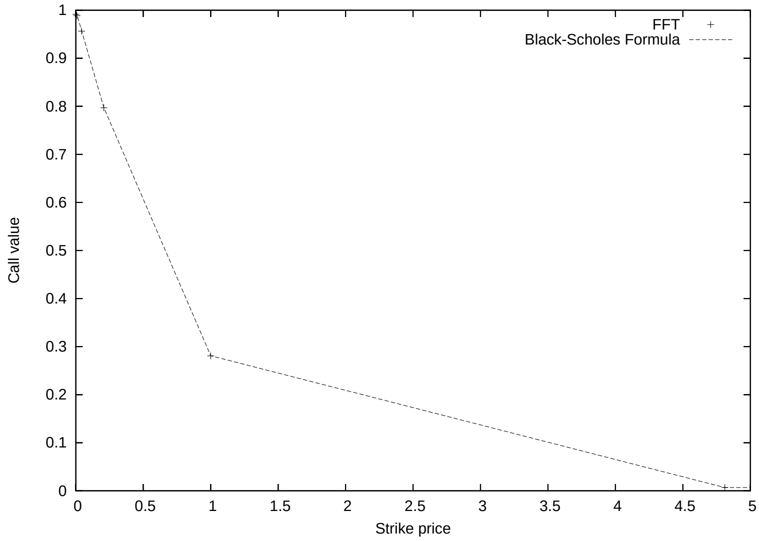

Figure 3.1: The call values in terms of the strikes for $N = 16$, $\alpha = 0.75$ and $S_0 = 1$

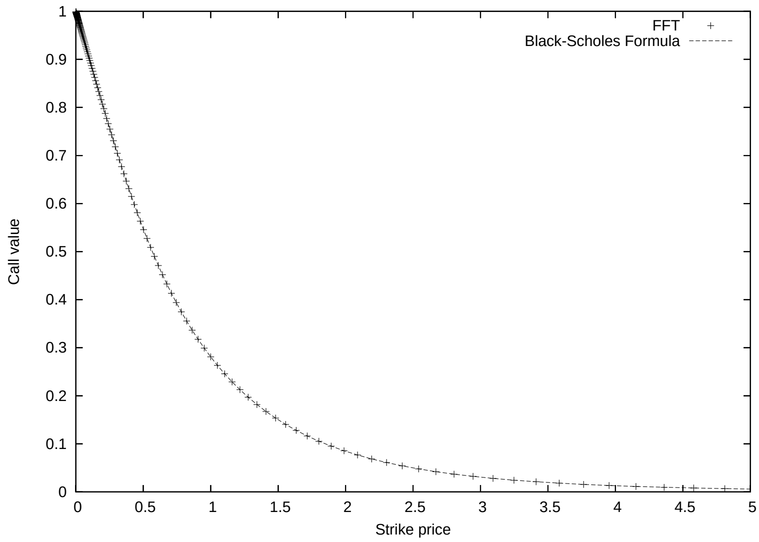

Figure 3.2: The call values in terms of the strikes for $N = 512$, $\alpha = 0.75$ and $S_0 = 1$

Comparison with the Black-Scholes formula

For comparison, we are going to use the analytic results from the Black-Scholes formula for the European initial call option price which is given by:

$$C_T(K) = S_0 N(d_1) - K\mathrm{e}^{-rT} N(d_2), \tag{3.8}$$

where N is the distribution function of the standard normal distribution $N(0,1)$ given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2} du. \tag{3.9}$$

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left\{ \log(S_0/K) + T(r + \frac{\sigma^2}{2}) \right\}, \tag{3.10}$$

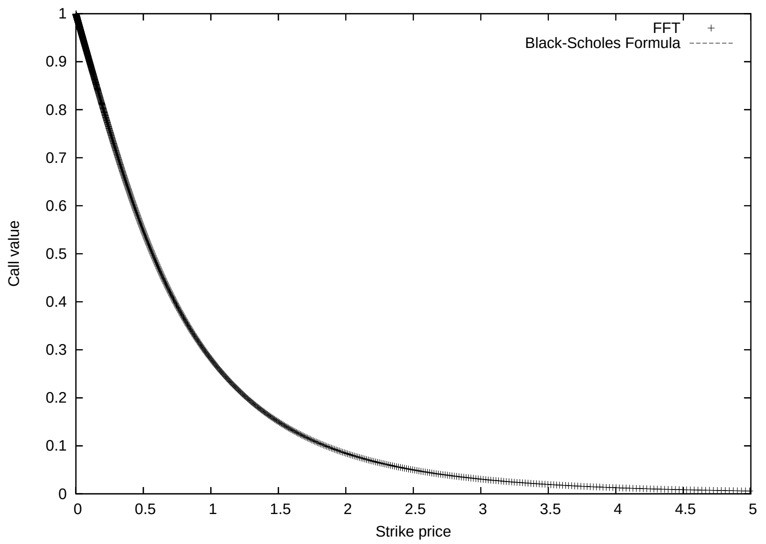

Figure 3.3: The call values in terms of the strikes for $N = 4096$, $\alpha = 0.75$ and $S_0 = 1$

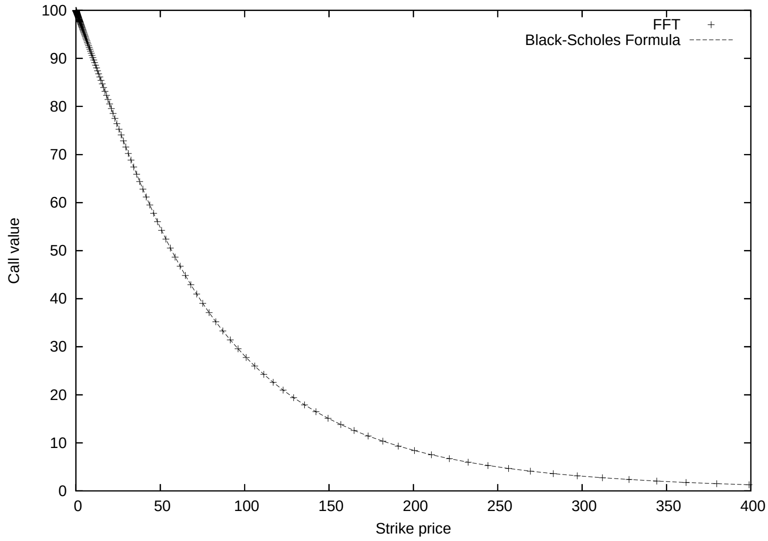

Figure 3.4: The call values in terms of the strikes for $N = 512$, $\alpha = 0.75$ and $S_0 = 100$

and

$$d_2 = d_1 - \sigma\sqrt{T}.$$

The proof of the Black-Scholes formula can be found in [Mar07].

The percentage differences between analytical and FFT prices is given by:

$$PAE = \frac{|\text{Analytical} - \text{FFT}|}{\text{Analytical}}, \tag{3.11}$$

where PAE stands for Percentage Absolute Error. It is given by Figure 3.5 for $N = 512$ and $S_0 = 100$. And it is shown that the method is more accurate near the initial price S_0 since we have constructed the sum for the FFT such that the strikes have been chosen around the initial price. And the error increases when the strikes go far away from this inital price.

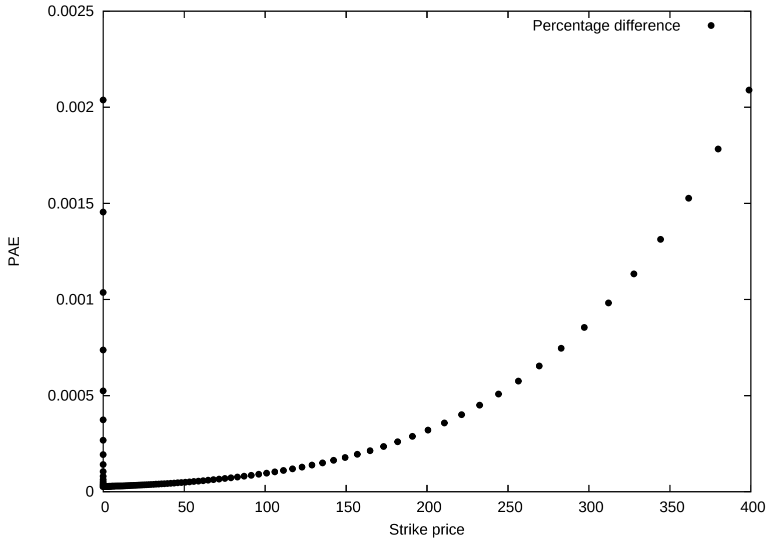

Figure 3.5: PAE given by Equation (3.2)

Comparison of the speed with the Monte Carlo simulation

The Monte Carlo simulation of a random variable X consists of considering M independent random variables X_i with the same distribution as X and expectation:

$$E(X) \approx \frac{1}{M} \sum_{i=1}^{M} X_i. \tag{3.12}$$

Then, for the European call option valuation, we have to compute equation (3.12) such that

$$X_i = \mathrm{e}^{-rT}(S_i - K)^+,$$

where S_i is the terminal price of the underlying asset obtained from the Black-Scholes model.

While executing the code, the fast computation of the prices by the FFT is justified compared with the Monte Carlo simulation. The table below shows the computing time of the both methods for different values of N.

N	Monte Carlo	FFT-method
512	111.31	0.03
1024	213.32	0.07

The Monte Carlo simulation was made for 10 different strikes, with only 10 repetitions.

Figure 3.6 shows the three methods in the same graph i.e the Monte Carlo simulation, the Black-Scholes formula and the FFT-method. The FFT method was simulated with $N = 512$ and the Monte Carlo with 10 repetitions and 512 time steps.

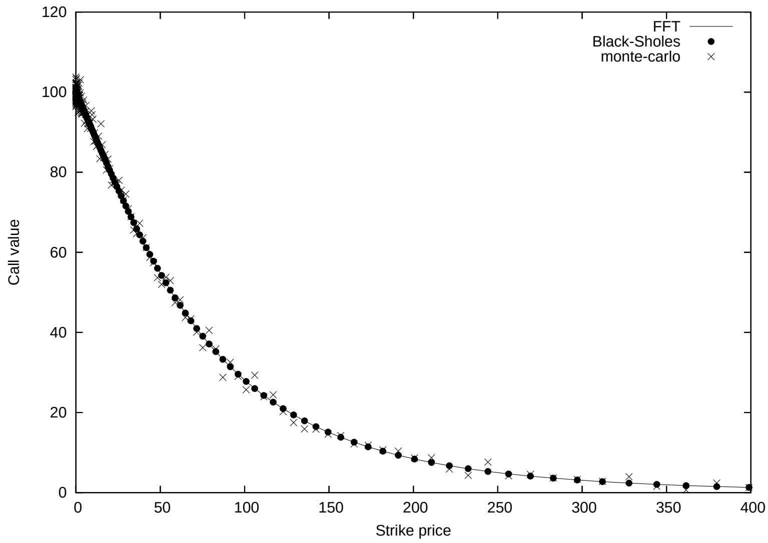

Figure 3.6: European call option prices obtain by Monte Carlo simulation, the Black-Scholes formula and the FFT method with initial price $S_0 = 100$.

Code to compute the call value using the FFT-method.

We use the python module for the fast Fourier transform.

```
# The characteristic function of the the log price at time t=0.
def kar(x,S_0,T,sigma,r):
    a=-(T*(sigma**2)*(x**2))/2
    b=(log(S_0)+(r-(sigma**2)/2)*T)*x
    z=complex(a,b)
    f=e**z
    return f

#The Fourier Transform of the call value function.
def Trans(u,S_0,T,sigma,r,alpha):
    v=complex(u,-(alpha+1))
    f=(e**(-r*T))*kar(v,S_0,T,sigma,r)
       /complex((alpha**2)+alpha-(u**2),(2*alpha+1)*u)
    return f

#Kronecker function.
def Kron(n):
    if n==0:
        f=1
    else:
        f=0
    return f

#The function to which we compute the FFT.
def X(N,b,S_0,T,sigma,r,alpha,h):
    X=[]
    for j in range(1,N+1):
        u=(j-1)*h
        x= e**(complex(0,b*u))*Trans(u,S_0,T,sigma,r,alpha)
             *(h/3)*(3+(-1)**j-Kron(j-1))
        X.append(x)
    return X


#FFT of the function X.
def fast(N,b,S_0,T,sigma,r,alpha,h):
    a=array(X(N,b,S_0,T,sigma,r,alpha,h))
    b=fft(a)
    return b
```

```
#The call value in terms of the strike K by the FFT method.
def plotB(N,S_0,T,sigma,r,alpha,h):
    gp=Gnuplot.Gnuplot(persist=1)
    Lambda=(2*pi)/(N*h)
    b=N*lamb/2
    C1=[]
    F=fast(N,b,S_0,T,sigma,r,alpha,h)
    for l in range(1,N+1):
        k=-b+Lambda*(l-1)
        K=e**k
        c1=(e**(-alpha*k)/pi)*F[l-1]
        if 0.01<=K<=400:
            C1.append([K,c1.real])
    return C1
```

Conclusion

Damping the call value function of the European option, we have succeeded in computing its Fourier transform. Numerically solving the analytical expression of the inverse Fourier gives us the value of the option. We use the FFT algorithm which rapidly computes the discrete Fourier transform. This fact has been demonstrated by implementing the Monte Carlo method of option pricing for comparison.

However, the choosing of the parameters and the algorithm steps need to be carefully studied in order to have accurate results. In fact, the freedom of choice of the damping coefficient α and the integration path h affect the accuracy of the method.

The method of Fourier transform works for many other models in financial market but the reason of the Black-Scholes model was to simplify the computation of the option prices. However, it is well agreed that some of the assumptions of this model are not justified in the market such as the constant volatility. One can analyse the FFT method deeper by considering some other realistic assumptions of the market and seeing how the choice of the numerical parameters can be made in each model.

Aknowledgments

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