Numerical Implementation of Adomian Decomposition Method for Volterra Integral Equations of the Second Kind with Weakly Singular Kernels

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Abstract

The objective of this essay is to determine the efficiency of the Adomian decomposition method by numerically using it to solve Volterra integral equations of the second kind with weakly singular kernels. We also compare the results of this method to the results of the numerical method implemented by using rational basis.

Contents

1. Introduction

Most of the phenomena that arises in real world are described by non linear differential and integral equations. However, most of the methods developed in mathematics are usually used in solving linear differential and integral equations. The recently developed decomposition method proposed by American mathematician, George Adomian(1923-1996), have been receiving much attention in recent years in applied mathematics in general [Meh06] . The Adomian decomposition method has the adventage of converging to the exact solution, and can easily handle a wide class of both linear and non linear differential and integral equations.

In this essay, we begin by giving a general introduction to Adomian decomposition method. We then discuss the Volterra integral equations and existence of its solution in chapter 2.The Numerical implementation of Adomian decomposition method to Volterra integral equation is examined in chapter 3. We conclude by implementing the Adomian decomposition method to solve Volterra integral equations of the second kind with weakly singular kernels.

1.1 General concepts of Adomian decomposition method

Consider the equation

$$Fu(t) = g(t) \ , \tag{1.1}$$

where F is a linear combination of linear and non linear operators. The method [Geo94] consists of decomposing the solution u into sum of an infinite number of components defined by the decomposition series:

$$u = \sum_{n=0}^{\infty} u_n \ , \quad \text{where the } u_n\text{'s are calculated recursively .} \tag{1.2}$$

To have an overview of the method, we consider the cubic Schrödinger equation:

$$\frac{\partial u}{\partial t} - i\frac{\partial^2 u}{\partial x^2} - iq\left|u^2\right| u = 0, \tag{1.3}$$

where by identification to (1.1) $Fu = \dfrac{\partial u}{\partial t} - i\dfrac{\partial^2 u}{\partial x^2} - iq\left|u^2\right| u$, and $g = 0$, with the boundary conditions: $u(x, 0) = f(x)$.

If we want to integrate equation (1.3) as a function of time t, the decomposition method requires us to denote:

$\dfrac{\partial u}{\partial t} = Lu$, $\quad -i\dfrac{\partial^2 u}{\partial x^2} = Ru$, and $Nu = -iq\left|u^2\right| u$, which implies that $F = L + R + N$.

The term L is the higher differential linear operator in t, and is assumed to have an inverse L^{-1}. In this case L^{-1} is a one fold integral operator, which means that:

$$L^{-1} = \int_0^t (.)\mathrm{d}s \ ;$$

hence

$$L^{-1}Lu(x,s) = u(x,t) - u(x,0) \implies u(x,t) = u(x,0) + L^{-1}g(t) - L^{-1}Ru(x,t) - L^{-1}Nu(x,t) \ . \tag{1.4}$$

The term R is called the remainder of the linear operator L. The term N is a non-linear function of u, and is assumed to have the following decomposition

$$Nu = \sum_{n=0}^{\infty} A_n \ .$$

Here, the A_n's are polynomials defined by the Adomain decomposition in the following way, [Geo94]

$A_0 = Nu_0,\quad u_0$ is equal to the sum of components which are not functions of u,
in this case $u_0 = u(x,0) + L^{-1}g(t)$,

$$A_1 = u_1 \left(\frac{\mathrm{d}N}{du}\right) u_0 \quad \text{and} \quad u_1 = -L^{-1}Ru_0 - L^{-1}A_0 \ ,$$

$$A_2 = 3u_2 \frac{\mathrm{d}N}{du} u_0 + \left(\frac{u^2}{2!}\right)\left(\frac{\mathrm{d}^2N}{du^2}\right) u_0 \quad \text{and} \quad u_2 = -L^{-1}Ru_1 - L^{-1}A_1$$

More generally ,

$$A_n = \sum_{1}^{n} c(j,n) \left(\frac{\mathrm{d}^{\mathrm{j}}N}{du^j}\right) u_0 \quad \text{and} \quad u_n = -L^{-1}Ru_{n-1} - L^{-1}A_{n-1} \ ,$$

where the $c(j,n)$ are products (or sums of products) of j components of u whose subscripts sum to n, divided by the factorial of the number of repeated subscripts [Geo94] .
As an example consider the non linear terms $Nu = u^n$ then the coefficients $c(j,n)$ are defined by

$$c(1,n) = u_n \quad \text{for} \quad j \geq 2 \quad c(j,n) = \sum_{i_1+i_2+\ldots+i_j=n} \frac{\prod_{k=1}^{k=j} u_{i_k}}{p} \ ,$$

with $0 < i_1 \leq i_2 \leq \ldots \leq i_j \leq n$. Also p is a product of factorial of α_{i_k} , where α_{ik} is the number of elements that are equal to u_{ik}. Consider the following example.

$$\text{If} \quad \prod u_{i_k} = u_1u_2u_3 \qquad \text{then} \qquad p = 1! \times 1! \times 1!,$$

$$\text{or if} \quad \prod u_{i_k} = u_1u_1u_1u_2u_3u_3 \qquad \text{then} \qquad p = 3! \times 1! \times 2! \ .$$

Example 1.1.1 *If the non linear term is $Nu = u^2$, then*

$$A_0 = u_0^2 \ , \quad A_1 = 2u_0u_1 \ , \quad A_2 = u_1^2 + 2u_0u_2$$
$$A_3 = 2u_1^2 + 2u_0^3 \ , \quad \textit{and} \quad A_4 = u_2^2 + 2u_1^3 + 2u_0^4 \ .$$

Example 1.1.2 *We now consider the first order differential equation without non linear term.*

$$y' + 2ty = 0 \quad \textit{with} \quad y(0) = 1.$$

The linear operator is $L = \frac{\mathrm{d}}{dx}(.) \implies L^{-1} = \int_0^x (.)\mathrm{d}t$ *and* $R = 2t(.)$ *with* $N = 0$.

The equation then yields that

$$Ly = -Ry \implies L^{-1}Ly = -L^{-1}Ry \implies y = y(0) - L^{-1}Ry \ .$$

We then have that

$$y_0(x) = y(0) = 1 \ , \quad y_1(x) = -L^{-1}Ry_0(t) = -\int_0^x 2t\mathrm{d}t = -x^2 \ ,$$

$$y_2(x) = -L^{-1}Ry_1(t) = -\int_0^x (2t)(-t^2)\mathrm{d}t = \frac{x^4}{2} \ ,$$

$$y_3(x) = -L^{-1}Ry_2(t) = -\int_0^x (2t)(\frac{t^4}{2})\mathrm{d}t = -\frac{x^6}{6} = -\frac{x^6}{3!} \ .$$

By continuing the iteration, we find that $y_n(x) = (-1)^n \frac{x^{2n}}{n!}$, *which implies that*

$$y(x) = \sum_{n=0}^{n=\infty} y_n(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \ldots = \exp(-x^2) \ .$$

Example 1.1.3 *Let us now consider the first order differential equation involving a non linear term.*

$$y' = y^2 - y \qquad \textit{with} \qquad y(0) = 2 \ .$$

We then have that

$$Ly = \frac{\mathrm{d}y}{dx}(.) \implies L^{-1}y(x) = \int_0^x (y(t))\,\mathrm{d}t \ , \quad Ry = -y \ , \quad Ny = y^2,$$

then

$$y = y(0) + L^{-1}Ny + L^{-1}Ry \ .$$

Also,

$$y_0(x) = 2 \implies A_0 = Ny_0 = 4$$

$$y_1(x) = L^{-1}Ry_0(t) + L^{-1}A_0 = \int_0^x -y_1\mathrm{d}t + \int_0^x -A_0\mathrm{d}t$$

$$= 2x \ .$$

As a result,

$$\begin{aligned} A_1(x) &= y_1 \left(\frac{\mathrm{d}N}{dy}\right) y_0 = 2y_1y_0 \\ &= 8x, \\ y_2(x) &= L^{-1}Ry_1(t) + L^{-1}A_1(t) = \int_0^x -y_1(t)\mathrm{d}t + \int_0^x A_1(t)\mathrm{d}t \\ &= 3x^2, \end{aligned}$$

In the same way,

$$\begin{aligned} A_2 &= 3y_2\frac{\mathrm{d}N}{dy}y_0 + \left(\frac{y^2}{2!}\right)\left(\frac{\mathrm{d}^2N}{dy^2}\right) y_0 = y_1^2 + 2y_0y_2 \\ &= 16x^2, \\ y_3(x) &= L^{-1}Ry_2(t) + L^{-1}A_2(t) \\ &= \int_0^x -y_2(t)\mathrm{d}t + \int_0^x A_2(t)\mathrm{d}t \\ &= \frac{13x^3}{3}. \end{aligned}$$

We can then make an approximation to the solution of order three:

$$y(t) \simeq 2 + 2t + 3t^2 + \frac{13t^3}{3} \ .$$

2. Volterra integral equations

2.1 Generality

2.1.1 Integral equations

Definition 2.1.1 *An integral equation is generally defined as an equation which involves the integral of an unknown function. A linear (non-linear) integral equation is an integral equation which involves a linear (non linear) expression of the unknown function.*

Example 2.1.1 *The following are linear integral equations:*

$$\cos(x) = \int_a^b \log(x-t)y(t)\mathrm{d}t \ ,$$

$$y(t) = t^2 - \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} y(s)\mathrm{d}s, \qquad t \in [0,1] \ .$$

These are non-linear integral equations:

$$x^3 + 4x = \int_a^b \log(x-t)y^2(t)\mathrm{d}t \ ,$$

$$y(t) = t^{\frac{1}{2}} + \frac{3}{8}\pi t^2 - \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \ln(y(s))\mathrm{d}s, \qquad t \in [0,3] \ .$$

2.1.2 Fredholm integral equations

Definition 2.1.2 *Let Ω be a measurable set in a measurable space E, and let μ be a positive measure defined on E. Fredholm integral equations are integral equations divided into two groups, referred to as Fredholm Integral equations of the first or the second kind. They have the following general expression:*

$$\int_\Omega K(x,t,u(t))\,\mathrm{d}\mu(t) = F(x,u(t)) \ . \tag{2.1}$$

Where K and F are known functions. K is called the kernel of the integral equation. In this essay we are only interrested in the Fredholm integral equations of the following form

$$a(x)u(x) - \int_\Omega K(x,t,u(t))\,\mathrm{d}\mu(t) = f(x) \ . \tag{2.2}$$

They satisfy the following two conditions:

- *a, K and f are known functions.*
- *u is an unknown function to be determined.*

If $a(x) = 0$ for any x the equation is a Fredholm integral equation of the first kind.
If $a(x) \neq 0$ the equation can be written as:

$$u(x) = f(x) + \int_{\Omega} K(x, t, u(t)) \, \mathrm{d}\mu(t) \tag{2.3}$$

and it is said to be of the second kind.
Note that : if $K(x, t, u(t)) = K'(x, t)\, g(u(t))$ where g is a known function of u, K' is also called the kernel of the integral equation.

In this essay, we are interested only in the case when Ω is a finite interval $[a, b] \subseteq \mathbb{R}$. Hence $\mu(t)$ becomes a Lebesgue measure defined by $\mathrm{d}\mu(t) = \mathrm{d}t$. In this case, the Fredholm integral equations of the first and second kind will respectively have the following expressions:

$$f(x) = \int_{a}^{b} K(x, t, u(t)) \, \mathrm{d}t, \qquad x \in [a, b] \ , \tag{2.4}$$

$$u(x) = f(x) + \int_{a}^{b} K(x, t, u(t)) \, \mathrm{d}t, \qquad x \in [a, b] \ . \tag{2.5}$$

2.2 Volterra integral equations

Definition 2.2.1 *In the space of real numbers $\mathbb{R}$ (of which we are interested) the Volterra integral equations are a particular case of Fredholm integral equations, where the kernel $K(x, t)$ vanishes for t $\geq x$. They are divided into two groups referred to as the first and second kind. The Volterra integral equations of the first kind are of the form*

$$f(x) = \int_{a}^{x} K(x, t, u(t)) \, \mathrm{d}t, \quad x \in [a, b] \ . \tag{2.6}$$

Those of the second kind are of the form

$$u(x) = f(x) + \int_{a}^{x} K(x, t, u(t)) \, \mathrm{d}t \quad x \in [a, b]. \tag{2.7}$$

In these two cases u, $f(x)$ and K satisfy the same conditions as in definition 2.1.2 .

Example 2.2.1 *The following is a Volterra integral equation of the second kind :*

$$y(x) = 1 - \int_0^x \sin(x - y(t))\,\mathrm{d}t, \qquad 0 \le x \le 1 \ .$$

This is a Volterra integral equation of the first kind :

$$\sin(x) + \int_2^x \sin(x-t)y^3(t)\mathrm{d}t = 0, \qquad 2 \le x \le 5 \ .$$

Definition 2.2.2 *A weakly singular kernel is a kernel of the form :*

$$K(x,t,y(t)) = \frac{H(x,t,y(t))}{(x-t)^\alpha} \ , \tag{2.8}$$

where $H(x,t,y(t))$ is a continuous function of t and x such that x and t belongs to $[a,b]$, and $0 < \alpha < 1$. A Volterra integral equation with the kernel in this form is said to be Volterra integral equation of the second kind with a weakly singular kernel.

Example 2.2.2 *The Abel integral equation,*

$$\int_a^x \frac{1}{(x-t)^{\frac{1}{2}}} y(t)\mathrm{d}t = \sin(x) \ , \tag{2.9}$$

is an example of a Volterra integral equation of the second kind with a weakly singular kernel.

2.3 Uniqueness and existence of solution

Before solving an integral equation, we need some tools to determine the solvability of the problem. Here, the basic idea is to show first that any linear Volterra integral equation with a weakly singular kernel has a unique solution.
Note that , in this essay all the functions that we are dealing with belong to the Hilbert space $H = L_2[a,b]$, where $[a,b]$ is a finite interval include in $\mathbb{R}$.

Definition 2.3.1 *Let $f_1,\ f_2$ be two elements of H. The inner product on H is defined by*

$$\langle f_1, f_2 \rangle = \int_a^b f_1(x)\bar{f}_2(x)\mathrm{d}x \ .$$

The norm of a given function f in H is :

$$||f|| = \left(\int_a^b |f(x)|^2\,\mathrm{d}t \right)^{\frac{1}{2}} \ .$$

Definitions 2.3.1 *1. Consider the Fredholm integral equation of 2.5. The applications T and A defined by:*

$$\begin{array}{rccl} T: & H & \longrightarrow & H \\ & u & \longmapsto & Tu \quad \textit{such that ,} \end{array}$$

$$Tu(x) = f(x) + \int_a^b K\left(x, t, u(t)\right) \mathrm{d}t \ , \quad \textit{for any } x \in [a, b];$$

$$\begin{array}{rccl} A: & H & \longrightarrow & H \\ & u & \longmapsto & Au \quad \textit{such that ,} \end{array}$$

$$Au(x) = \int_a^b K\left(x, t, u(t)\right) \mathrm{d}t \ , \quad \textit{for any } x \in [a, b];$$

are operators on H. And the equation 2.7 can be expressed in the following form:

$$Tu = u = f + Au \ ,$$

u is called a fixed point of T.

2. The operator T is said to be contraction operator if there exists a constant α, such that

$$\|Tf_1 - Tf_2\| \leq \alpha \|f_1 - f_2\| \qquad 0 \leq \alpha < 1 \ , \tag{2.10}$$

for f_1 and f_2 elements of H.

Theorem 2.3.1 *[Har73] Let T be a continuous contraction operator on H. The equation*

$$Tf = f \tag{2.11}$$

has a unique solution .

Proof 2.3.1 *We first prove the existence of f; let $f_0 \in L_2[a, b]$ and defined the sequence f_n by*

$$f_{n+1} = Tf_n.$$

This sequence is a Cauchy sequence. To see this, we have that

$$\|f_{n+1} - f_n\| = \|Tf_n - Tf_{n-1}\| \leq \alpha \|f_n - f_{n-1}\| \ ;$$

therefore,

$$\|f_{n+1} - f_n\| \leq \alpha^n \|f_1 - f_0\| \ .$$

Thence , for $m > n > 0$

$$\begin{aligned}\|f_m - f_n\| &= \|(f_m - f_{m-1}) + (f_{m-1} - f_{m-2}) + \ldots + (f_{n+1} - f_n)\| \\ &\leq \|(f_m - f_{m-1})\| + \|(f_{m-1} - f_{m-2})\| + \ldots + \|(f_{n+1} - f_n)\| \\ &= \alpha^{m-1}\|f_1 - f_0\| + \alpha^{m-2}\|f_1 - f_0\| + \ldots + \alpha^n\|f_1 - f_0\| \\ &= (\alpha^{m-1} + \alpha^{m-2} + \ldots + \alpha^n)\|f_1 - f_0\| \\ &= \alpha^n \frac{1-\alpha^{m-n}}{1-\alpha}\|f_1 - f_0\| \\ &\leq \alpha^n \frac{1}{1-\alpha}\|f_1 - f_0\| \ .\end{aligned}$$

Therefore,

$$\lim_{n\to+\infty} \|f_m - f_n\| = 0 \ .$$

By completeness of the Hilbert space H*, the sequences* (f_n) *converges to a function* f*. Thus by the continuity of* T*, we have that*

$$\lim_{n\to+\infty} Tf_n = \lim_{n\to+\infty} f_{n+1} = T \lim_{n\to+\infty} f_n \Rightarrow Tf = f \ .$$

Also note that

$$Tf_n = T(Tf_{n-1}) = T^n f_0 \Rightarrow \lim_{n\to+\infty} T^n f_0 = f \ .$$

Let us prove the uniqueness of the solution. Suppose that there exists f *and* g *such that* $Tf = f$ *and* $Tg = g$*. As* T *is contraction operator, we then have that*

$$\|Tf - Tg\| = \|f - g\| \leq \alpha\|f - g\| \Rightarrow (1-\alpha)\|f - g\| \leq 0 \Rightarrow f = g \quad \blacksquare$$

Theorem 2.3.2 *[Har73] Let* T *be a continuous operator on H. Suppose that there exists* n *in* $\mathbb{N}$ *such that* T^n *is a contraction operator. Then the equation*

$$Tf = f \tag{2.12}$$

has a unique solution.

Proof 2.3.2 *By the proof of the previous Theorem, there exists an* f *in* H *such that*

$$T^n f = f \ ,$$

which implies that

$$T^{kn} f = f, \ for \ k \in \mathbb{N} \ .$$

By the same proof, if we take $f_0 = Tf$ *, we have that ,*

$$\lim_{k\to+\infty} (T^n)^k f_0 = f \qquad \textit{and} \qquad \lim_{k\to+\infty} T^{kn} f_0 = \lim_{k\to+\infty} (T^{kn})Tf = \lim_{k\to+\infty} TT^{kn} f = Tf \ .$$

So that

$$Tf = f. \ \blacksquare$$

Lemma 2.3.1 *Let A and B be two operators such that*

$$A\phi(x) = \int_a^b K_1(x,y)\phi(y)\mathrm{d}y$$

and

$$B\phi(x) = \int_a^b K_1^{'}(x,y)\phi(y)\mathrm{d}y \ .$$

Then AB is also an operator with the kernel

$$K_2(x,y) = \int_a^b K_1(x,z)K_1^{'}(z,y)\phi(y)\mathrm{d}y \ .$$

And if K_1 and $K_1^{'}$ are bounded, it means that there exists $M_1 \in \mathbb{N}$ and $M_2 \in \mathbb{N}$ such that

$$|K_1(x,y)| \leq M_1 \quad \text{and} \quad |K_1^{'}(x,y)| \leq M_2 \ .$$

Then for any $x, \ y$ elements of $[a,b]$, K_2 is also bounded with

$$|K_2(x,y)| \leq M_1 M_2 (b-a) \ .$$

Here we have defined the product AB by:

$$AB\phi(x) = A(B\phi)(x) = \int_a^b K_1(x,y)\left(B\phi(y)\right)\mathrm{d}y \ .$$

Proof 2.3.3 *We have that*

$$\begin{aligned} AB\phi(x) = A\left(B\phi(x)\right) &= \int_a^b K_1(x,z)B\phi(z)\mathrm{d}z \\ &= \int_a^b K_1(x,z)\left(\int_a^b K_1^{'}(z,y)\phi(y)\mathrm{d}y\right)\mathrm{d}z \\ &= \int_a^b \left(\int_a^b K_1(x,z)K_1^{'}(z,y)\mathrm{d}z\right)\phi(y)\mathrm{d}z \end{aligned}$$

It shows that AB is an operator with the kernel K_2, and

$$K_2(x,y) = \int_a^b K_1(x,z)K_1^{'}(z,y)\mathrm{d}z \ ,$$

so that

$$|K_2(x,y)| \leq \int_a^b |K_1(x,z)K_1^{'}(z,y)|\mathrm{d}z \leq M_1M_2(b-a) \ . \quad \blacksquare$$

Lemma 2.3.2 *If A is an operator with the kernel K, such that*

$$|K(x,y)| \leq M \quad \textit{and} \quad A\phi(x) = \int_a^b K(x,y)\phi(y)\mathrm{d}y \ ,$$

then A^n for $n \in \mathbb{N}$ is also an operator with the kernel K_n, with

$$K_n(x,y) = \int_a^b K(x,z)K_{n-1}(z,y)\mathrm{d}z \ .$$

If T is an operator on H with

$$T\phi = f + A\phi \qquad f \in H \ ,$$

then T is continuous

Proof 2.3.4 *By lemma 2.3.1, A^2 is an operator with the kernel K_2 such that,*

$$K_2(x,y) = \int_a^b K(x,z)K_1(z,y)\mathrm{d}z \quad \textit{where} \quad K_1(x,y) = K(x,y) \ .$$

Now suppose that A^n is an operator with the kernel K_n such that

$$K_n(x,y) = \int_a^b K(x,z)K_{n-1}(z,y)\mathrm{d}z$$

since $AA^n = A^{n+1}$, we see by lemma 2.3.1 that A^{n+1} is an operator with the kernel K_{n+1} such that

$$K_{n+1}(x,y) = \int_a^b K(x,z)K_n(z,y)\mathrm{d}z$$

To prove the continuity of the operator T, we have that

$$\|T\phi_1 - T\phi_2\| = \left| \int_a^b T\left(\phi_1(y) - \phi_2(y)\right) \overline{T\left(\phi_1(y) - \phi_2(y)\right)}\mathrm{d}y \right|$$

$$= \int_a^b |T\phi_1(y) - \phi_2(y)|^2\mathrm{d}y \ ,$$

so that

$$|T\phi_1(y) - T\phi_2(y)|^2 = \left| \int_a^b k(y,s)\,(\phi_1(s) - \phi_2(s))\,\mathrm{d}s \right|^2$$
$$\leq \left(\int_a^b |K(y,s)\,(\phi_1(s) - \phi_2(s)|)\,\mathrm{d}s \right)^2 ,$$

which means that

$$|T\phi_1(y) - T\phi_2(y)|^2 \leq \left[\int_a^b |K(y,s)|^2 \mathrm{d}s \right] \left[\int_a^b |\phi_1(s) - \phi_2(s)|^2 \mathrm{d}s \right]$$
$$\leq M^2(b-a)\|\phi_1 - \phi_2\| ,$$

then

$$\|T\phi_1 - T\phi_2\| \leq M^2(b-a)\|\phi_1 - \phi_2\| .$$

Hence T is continuous on H. ■

Theorem 2.3.3 *[Har73] Let $f \in L_2[0,1]$, and let K be a kernel such that*

$$\int_0^1 |K(x,y)|^2 \mathrm{d}x\mathrm{d}y \leq \infty .$$

Then, the Volterra integral equation

$$\phi(x) = f(x) + \int_0^x K(x,y)\phi(y)\mathrm{d}y \qquad x \in [0,1] , \tag{2.13}$$

has a unique solution in $L_2[0,1]$.

Lemma 2.3.3 *Consider the operator A with the kernel K defined as in the Theorem 2.3.3. Then the kernel K_n associate to the operator A^n satisfy the following inequality.*

$$|K_n(x,y)|^2 \leq \frac{M^n(x-y)^{n-2}}{(n-2)!} \qquad \text{for} \quad n \geq 2 ,$$

Proof 2.3.5 *By lemma 2.3.2, we have that*

$$K_n(x,y) = \int_0^x K(x,z)K_{n-1}(z,y)\mathrm{d}z$$

but

$$\begin{cases} K_{n-1}(z,y) = 0 & \text{for } \ y \geq z \\ K(x,z) = 0 & \text{for } \ z \geq x. \end{cases}$$

Hence

$$K_n(x,y) = \int_y^x K(x,z)K_{n-1}(z,y)\mathrm{d}z \quad 0 \leq y \leq x \leq 1 \ ,$$

for $n = 2$ we have that

$$\begin{aligned} |K_2(x,y)|^2 &= |\int_y^x K(x,z)K_1(z,y)\mathrm{d}z|^2 \\ &\leq \int_y^x |K(x,z)|^2\mathrm{d}z \int_y^x |K_1(z,y)|^2\mathrm{d}z \end{aligned}$$

By hypothese K is square integrable which mean that there exist $M \in \mathbb{N}$ such that , $\int_y^x |K(x,y)|^2\mathrm{d}y \leq M$. Thus

$$|K_2(x,y)|^2 \leq M^2 \ .$$

suppose that it is true for $n \geq 2$. Then

$$\begin{aligned} |K_{n+1}(x,y)|^2 &= |\int_y^x K(x,z)K_n(z,y)\mathrm{d}z|^2 \\ &\leq \int_y^x |K(x,z)|^2\mathrm{d}z \int_y^x |K_n(z,y)|^2\mathrm{d}z \\ &\leq M \int_y^x \frac{M^n(z-y)^{n-2}}{(n-2)!}\mathrm{d}z = \frac{M^{n+1}}{(n-2)!}\int_y^x (z-y)^{n-2}\mathrm{d}z \\ &= \frac{M^{n+1}}{(n-2)!}\left[\frac{(z-y)^{n-1}}{n-1}\right]_y^x = \frac{M^{n+1}(x-y)^{n-1}}{(n-1)!} \ . \end{aligned}$$

So the assertion is true $\forall n \geq 2$. ■

Now we can prove the Theorem 2.3.3

Proof 2.3.6 *Let us define the operators T and A such that*

$$T\phi(x) = f(x) + \int_0^x K(x,y)\phi(y)\mathrm{d}y \quad \text{and}$$

$$A\phi(x) = \int_0^x K(x,y)\phi(y)\mathrm{d}y \ , \qquad \text{for } \ 0 \leq x \leq 1 \ .$$

it follows that

$$T\phi = f + A\phi \quad \Longrightarrow \quad T^2\phi = T(T\phi) = f + Af + A^2\phi \ .$$

More generally

$$T^n\phi = T(T^{n-1}\phi) = f + Af + A^2 f + ... + A^n\phi \ .$$

Let us prove the following assertion. There exists n in $\mathbb{N}$ such that T^n is a contraction operator

$$\begin{aligned} |T^n\phi_1(x) - T^n\phi_2(x)|^2 &= \left| \int_0^x K_n(x,y)(\phi_1(y) - \phi_2(y))\mathrm{d}y \right|^2 , \\ &\le \left(\int_0^x |K_n(x,y)(\phi_1(y) - \phi_2(y))|\mathrm{d}y \right)^2 \\ &\le \int_0^x |K_n(x,y)|^2\mathrm{d}y \int_0^x |\phi_1(y) - \phi_2(y)|^2\mathrm{d}y. \end{aligned}$$

Therefore, by lemma 2.3 we have that

$$\begin{aligned} |T^n\phi_1(x) - T^n\phi_2(x)|^2 &\le \frac{M^{n-1}}{(n-2)!} \int_0^x (x-y)^{n-2}\mathrm{d}y \, ||\phi_1 - \phi_2|| \\ &\le \frac{M^n x^{n-1}}{(n-1)!} \, ||\phi_1 - \phi_2|| \, , \end{aligned}$$

then we have that

$$||T^n\phi_1 - T^n\phi_2||^2 \le \frac{M^n}{(n-1)!} \, ||\phi_1 - \phi_2||$$

however, for large n

$$\frac{M^n}{(n-1)!} < 1 \ .$$

Thus there exists $n \in \mathbb{N}$ such that T^n is a contraction operator.
Hence the equation

$$T\phi = \phi$$

has a unique solution. ■

Theorem 2.3.4 *[Har73] Let $f \in L_2[0,1]$, then the Volterra integral equation with weakly singular kernel*

$$\phi(x) = f(x) + \int_0^x K(x,y)\phi(y)\mathrm{d}y \ , \tag{2.14}$$

where

$$K(x,y) = \frac{H(x,y)}{(x-y)^\alpha}, \qquad 0 < \alpha < 1$$

and H a continuous functions in $[0,1]$, has a unique solution .

Proof 2.3.7 *If H is continuous in $[0,1]$, there exists $M \in \mathbb{N}$ such that*

$$|H(x,y)| \leq M \quad \forall x,y \text{ in } [0,1].$$

Let

$$K_2(x,y) = \int_y^x \frac{H(x,z)}{(x-z)^\alpha}\frac{H(z,y)}{(z-y)^\alpha}\mathrm{d}z \quad \text{and } z = y + (x-y)u \ .$$

We have that

$$K_2(x,y) = \int_0^1 \frac{H(x,y+(x-y)u)}{(x-y)^\alpha(1-u)^\alpha}\frac{H(y+(x-y)u,y)}{(x-y)^\alpha u^\alpha}(x-y)\mathrm{d}u,$$

then

$$|K_2(x,y)| \leq \frac{M^2}{(x-y)^{2\alpha-1}}\int_0^1 \frac{1}{u^\alpha(1-u)^\alpha}\mathrm{d}u \ ,$$

and

$$\int_0^1 \frac{1}{u^\alpha(1-u)^\alpha}\mathrm{d}u = \int_0^\epsilon \frac{1}{u^\alpha(1-u)^\alpha}\mathrm{d}u + \int_\epsilon^1 \frac{1}{u^\alpha(1-u)^\alpha}\mathrm{d}u \ .$$

The function

$$\frac{1}{u^\alpha(1-u)^\alpha}$$

is integrable in any interval of the form

$$(\epsilon^{'},\epsilon) \ \subset \ (0,1) \qquad \text{with} \qquad 0 < \epsilon^{'} < \epsilon < 1 \ .$$

This function becomes equivalent to the the integrable funtions,

$$\frac{1}{u^\alpha} \quad \text{and} \quad \frac{1}{(1-u)^\alpha}$$

as u approaches zero and u approaches to 1 respectively .
Therefore,

$$\int_0^1 \frac{1}{u^\alpha(1-u)^\alpha}\mathrm{d}u$$

is integrable in $[0,1]$. This means that there exists $M \in \mathbb{N}$ such that

$$|K_2(x,y)| \leq \frac{M}{(x-y)^{2\alpha-1}} \ .$$

For $0 \leq \alpha \leq \frac{1}{2}$, $(2\alpha-1) \leq 0$ this implies that K_2 is square integrable. Hence by Theorem 2.3.2, the equation has a unique solution.
Now for $\frac{1}{2} < \alpha < 1$, suppose that for $n \geq 2$ there exist $M_1 \in \mathbb{N}$ such that

$$|K_n(x,y)| \leq \frac{M_1}{(x-y)^{n\alpha-(n-1)}} \ ;$$

we the have that

$$|K_{n+1}(x,y)| = \left| \int_y^x \frac{H(x,z)}{(x-z)^\alpha} K_n(z,y) \mathrm{d}z \right|$$
$$\leq M \int_y^x \left| \frac{H(x,z)}{(x-z)^\alpha} \frac{1}{(z-y)^{n\alpha-(n-1)}} \right| \mathrm{d}z,$$
$$\leq \frac{MM_1}{(x-y)^{(n+1)\alpha-n}} \int_0^1 \frac{1}{u^{n\alpha-(n-1)}(1-u)^\alpha} \mathrm{d}u \ .$$

But for $n > \dfrac{1}{1-\alpha}$, *the inequality* $n\alpha - (n-1) < 0$ *holds. Therefore, the function*

$$\frac{1}{u^{n\alpha-(n-1)}(1-u)^\alpha}$$

is integrable in $[0,1]$. *Let* $M_2 \in \mathbb{N}$ *such that*

$$\int_0^1 \frac{1}{u^{n\alpha-(n-1)}(1-u)^\alpha} \mathrm{d}u \leq M_2 \ ;$$

letting $M^{'} = MM_1M_2$, *we have that*

$$|K_{n+1}(x,y)| \leq \frac{M^{'}}{(x-y)^{(n+1)\alpha-(n)}} \ .$$

Thus for any $n \geq 2$, *there exist* $M \in \mathbb{N}$ *such that*

$$|K_n(x,y)| \leq \frac{M}{(x-y)^{n\alpha-(n-1)}}$$

and for $n > \dfrac{1}{1-\alpha}$, *the function* $\dfrac{M}{(x-y)^{n\alpha-(n-1)}}$ *is continuous. Hence, there exists* $M \in \mathbb{N}$ *such that*

$$|K_n(x,y)| \leq M \ .$$

For such n, *let* $L = T^n$ *an operator on* H.
The kernel of L *is*

$$K^{'} = K_n \quad \Longrightarrow \quad K_2^{'} = K_{2n} \ .$$

Now K_{2n} *is bounded, which means that* $K_2^{'}$ *is square integrable .*
By Theorem 2.3.3, there exists $p \in \mathbb{N}$ *such that*

$$L^p = T^{pn}$$

is a contraction operator.
Finally by Theorem 2.3.2, our equation has a unique solution. ■

Now consider the non-linear Volterra integral equation of the second kind with weakly singular kernel,[S. 91]

$$y(t) = g(t) - \int_0^t \frac{1}{(t-s)^{\alpha}} K\left(t, s, y(s)\right) \mathrm{d}s \quad t \leq a \in \mathbb{R}^+ , \tag{2.15}$$

with $0 < \alpha < 1$, and $K\left(x, t, y(s)\right)$ is continuous with respect to s and t, and uniformly Lipschitz continuous with respect to y,

$$||K\left(t, s, y_1\right) - K\left(t, s, y_2\right)|| \leq L\,||y_1 - y_2||\,. \tag{2.16}$$

Proposition 2.3.1 *The equation 2.15 has a unique solution for small value of L.*

Proof 2.3.8 *Let us define the operator T, such that :*

$$Ty(x) = g(x) - \int_0^x H\left(x, s, y(s)\right) \mathrm{d}s \quad \textit{with} \;\; x \leq a \in \mathbb{R}^+,$$

with

$$H\left(t, s, y(s)\right) = (t-s)^{-\alpha} K\left(t, s, y(s)\right)$$

but

$$||K\left(t, s, y_1\right) - K\left(t, s, y_2\right)|| \leq L\,||y_1 - y_2||\,,$$

which implies that

$$||H\left(t, s, y_1\right) - H\left(t, s, y_2\right)|| \leq L(t-s)^{-\alpha}\,||y_1 - y_2|| \;\; ,$$

then

$$\begin{aligned}
|Ty_1(t) - Ty_2(t)| &\leq L\,||y_1 - y_2|| \int_0^t (t-s)^{-\alpha} \mathrm{d}t \\
&= L\,||y_1 - y_2||\,(-t)^{1-\alpha} \int_0^1 (1-u)^{-\alpha} \mathrm{d}u \\
&\leq \frac{t^{1-\alpha}}{1-\alpha} L\,||y_1 - y_2|| \;\; ,
\end{aligned}$$

then we have that

$$||Ty_1 - Ty_2|| \leq \frac{a^{1-\alpha}}{1-\alpha} L\,||y_1 - y_2|| \;\; .$$

It show that for L very small, such that $\frac{a^{1-\alpha}}{1-\alpha} L < 1$, T is a contraction operator. Hence the equation 2.15 have a unique solution. ∎

Note that the Proposition above holds for any positive real number L.

3. Numerical methods

Here we will outline some themes of numerical integration methods which constitute a broad family of algorithms for calculating the numerical value of a definite integral.

3.1 Interpolation method

The idea of this method is that, given the value of some unknown function at a number of points, what value does that function have at other points ? This means that if we have a function ϕ defined in a finite interval $[a, b]$ such that

$$y_i = \phi(x_i) \quad \text{for} \quad x_i \in [a, b] \qquad i = 0,\ 1\ \ldots\ \ n\ ,$$

by choosing an appropriate set of basic functions, $\phi_0,\ \phi_1,\ \ldots,\ \phi_n$, such that

$$\phi_i\left(x_j\right) = \begin{cases} 1 & \text{for} \qquad i = j \\ 0 & \text{for} \qquad i \neq j \end{cases}$$

for any $x \in [a, b]$, then $\phi(x)$ can be approximated by

$$\phi(x) \simeq \sum_{i=0}^{n} y_i \phi_i(x)\ . \tag{3.1}$$

The interpolation method is quick and easy, but it is not very precise.

3.2 Quadrature method

There are many algorithms used to approximate the numerical value of an integral equation. Most of them consist of approximating the definite integral

$$I = \int_a^b f(x)\mathrm{d}x\ , \qquad a \text{ and } b \text{ are finite numbers}$$

by a quadratic formula.This approximation assumes a form such as

$$I \simeq \sum_{i=0}^{n} w_i f(x_i)\ , \tag{3.2}$$

where $x_i \in [a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$, the nodes x_i and weights w_i depends on the type of method used to approximate I.

Example 3.2.1 *This example concerns the Euler method. Let us consider the following integral*

$$I = \int_a^b f(x)\mathrm{d}x \ , \quad \textit{where} \ \ a \in \mathbb{R} \ \textit{and also} \ b \in \mathbb{R} \ .$$

The method consits by considering the weight w_i*, such that* $w_i = h = \dfrac{b-a}{M}$*, where* $M \in \mathbb{N}$*. Then the integral* I *can be approximated by:*

$$I = h \sum_{i=0}^{M} f(ih) \ .$$

3.3 Rational basis method

Consider the rational functions [S. 91]

$$r_0(t) = \frac{h-t}{h+kt} \quad \text{and} \quad r_1(t) = 1 - r_0(t).$$

The variables t and h are such that $t \in [0,h] \subset [0,T]$, where t is a positive real number, and k is a parameter in the interval $(-1,\infty)$.
Let $t_0 < t_1 < t_2 < \cdots < t_n$ be a subdivision of the interval $[0,T]$, with $t_0 = 0$, $t_n = T$.
The rational basis functions, $\phi_0, \ \phi_1, \ldots, \phi_{n-1}, \ \phi_n$ are defined by $\phi_0(t) = r_0(t-t_0)$ and for $1 \leq j \leq n-1$

$$\phi_j(t) = \begin{cases} r_1(t-t_{j-1}) & \text{if} \ \ t_{j-1} \leq t < t_j \ , \\ r_0(t-t_j) & \text{if} \ \ t_j \leq t < t_{j+1} \ , \end{cases}$$

with $\phi_n(t) = r_1(t - t_{n-1})$.

3.3.1 Approximation of Volterra integral equation

Consider the Volterra integral equation with weakly singular kernel

$$y(t) = g(t) - \int_0^t \frac{H(t,s)}{(t-s)^\alpha} y(s)\mathrm{d}s \ .$$

For a fixed value t_i of t, we have the following approximation:

$$H(t_i,s)y(s) \simeq \sum_{j=0}^{i} H(t_i,t_j)\phi_j(s)y(t_j) \ .$$

Then

$$y(t_i) \simeq g(t_i) - \int_0^{t_i} \left(\sum_{j=0}^{i} H(t_i, t_j) \frac{\phi_j(s)}{(t_i - s)^\alpha} y(t_j) \right) \mathrm{d}s$$
$$= g(t_i) - \sum_{j=0}^{i} H(t_i, t_j) y(t_j) \int_0^{t_i} \frac{\phi_j(s)}{(t_i - s)^\alpha} \mathrm{d}s \ .$$

By letting $s = t_j + ph$ it follows that the integral $\int_0^{t_i} \frac{\phi_j(s)}{(t_i - s)^\alpha} \mathrm{d}s$ can be expressed as $h^{1-\alpha} w_{ij}$, where w_{ij} is the quadrature weight. We then have that

$$y(t_i) \left(1 + h^{1-\alpha} H(t_i, t_i)) w_{ii}\right) \simeq g(t_i) - \sum_{j=0}^{i} H(t_i, t_j) y(t_j) \int_0^{t_i} \frac{\phi_j(s)}{(t_i - s)^\alpha} \mathrm{d}s \ .$$

Let $c_{ii} = 1 + h^{1-\alpha} H(t_i, t_i) w_{ii}$, then the algorithm of approximation is given by

$$y(t_0) = g(0) \ ,$$
$$y(t_1) = \frac{g(1)}{c_{11}} - \frac{h^{1-\alpha}}{c_{11}} H(t_1, t_0) y(t_0) w_{i0} \ .$$

More generally, $y(t_i) = \frac{g(t_i)}{c_{ii}} - \frac{h^{1-\alpha}}{c_{ii}} \sum_{j=0}^{i-1} H(t_i, t_j) y(t_j) w_{ij}$.

3.4 Numerical implementation of Adomian decomposition method

Consider the Volterra integral equations of the second kind

$$u(x) = f(x) + \int_0^x k\left(x, t, u(t)\right) \mathrm{d}t \ . \tag{3.3}$$

It is similar to the transformed form of Adomian formula by taking

$$L^{-1} = \int_0^x (.) \mathrm{d}t \ , \qquad Ru(t) = K\left(x, t, u(t)\right) \ , \quad \text{and} \quad Nu = 0 \ .$$

We then have that

$$u = f + L^{-1} Ru$$

and u can be decomposed as a series, given by:

$$u_0 = f \ , \quad u_1 = L^{-1} R u_0 \ ,$$

and for $n \geq 1$

$$u_n = L^{-1} R u_{n-1} \ .$$

Theorem 3.4.1 *If the series of numerical implementation of Adomian decomposition method is convergent, then it converges to the exact solution.*

Proof 3.4.1 *We have that*

$$u_0 = f \quad \text{and for } n \geq \quad u_n = L^{-1} R u_{n-1} \quad \Longrightarrow$$

$$\sum_{p=0}^{N} u_p = f + \sum_{p=0}^{N-1} L^{-1} R u_p \ ,$$

if the series of general term u_n is convergent then

$$\begin{aligned} &\lim_{N \longrightarrow \infty} \sum_{p=0}^{N} u_p = f + \lim_{N \longrightarrow \infty} \sum_{p=0}^{N-1} L^{-1} R u_p = f + L^{-1} R \lim_{N \longrightarrow \infty} \sum_{p=0}^{N-1} u_p \\ \Longrightarrow \quad &\sum_{p=0}^{\infty} u_p = f + L^{-1} R \sum_{p=0}^{\infty} u_p \\ &\Longrightarrow \quad u = f + L^{-1} R u \ . \quad \blacksquare \end{aligned}$$

Theorem 3.4.2 *The solution of the Volterra integral equation 3.3 by the Adomian decomposition method is unique.*

Proof 3.4.2 *Let*

$$u = \sum_{n=0}^{\infty} u_n \quad \text{and} \quad v = \sum_{n=0}^{\infty} v_n, \quad \text{be two solutions of equation } 3.3 \ ,$$

By hypothese we have that: $u_0 = f$ *and also* $v_0 = f \quad \Longrightarrow \quad u_0 = v_0$ *, suppose that it is true for a given* n*, such that* $n \geq 0$*, which means that* $u_n = v_n$ *. But*

$$u_{n+1}(x) = \int_0^x K\left(x, t, u_n(t)\right) \mathrm{d}t \quad \text{and} \quad v_{n+1}(x) = \int_0^x K\left(x, t, v_n(t)\right) \mathrm{d}t,$$

which implies that $u_{n+1}(x) = v_{n+1}(x) \quad$ *for any* $x, \quad$ *thus* $\quad u = v \ . \quad \blacksquare$

Example 3.4.1 *Consider the integral equation*

$$u(x) = x + \int_0^x (t-x)u(t)\mathrm{d}t \ .$$

We then have that

$$\begin{aligned}
u_0(x) &= x, \\
u_1(x) &= L^{-1}Ru_0(t) = \int_0^x (t-x)t\mathrm{d}t = \left[\frac{t^2}{2}(t-x)\right]_0^x - \int_0^x \frac{t^2}{2}\mathrm{d}t \\
&= \frac{-x^3}{3!}, \\
u_2(x) &= L^{-1}Ru_1(t) = \int_0^x (t-x)\frac{-t^3}{3!}\mathrm{d}t = \frac{-1}{3!}\left[\frac{t^4}{4}(t-x)\right]_0^x + \frac{1}{3!}\int_0^x (t-x)\frac{t^4}{4}\mathrm{d}t \\
&= \frac{x^5}{5!},
\end{aligned}$$

suppose that

$$u_n(x) = (-1)^n \frac{x^{2n+1}}{(2n+1)!} \qquad n \geq 1 \ ,$$

we then have that

$$\begin{aligned}
u_{n+1}(x) &= L^{-1}Ru_n(t) = \frac{(-1)^n}{(2n+1)!}\int_0^x (t-x)t^{2n+1}\mathrm{d}t \\
&= \frac{(-1)^n}{(2n+1)!}\left[\frac{t^{2n+2}}{(2n+2)!}(t-x)\right]_0^x - \frac{(-1)^n}{(2n+1)!}\int_0^x \frac{t^{t^{2n+2}}}{2n+2}\mathrm{d}t \\
&= \frac{(-1)^{n+1}}{(2n+3)!}x^{2n+3} = \frac{(-1)^{n+1}}{(2(n+1)+1)!}x^{(2(n+1)+1)} \ .
\end{aligned}$$

Thus for any $n \geq 1$

$$u_n(x) = (-1)^n \frac{x^{2n+1}}{(2n+1)!} \ .$$

Hence

$$u(x) = \sum_{n=0}^{\infty} u_n = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots = \sin(x) \ .$$

One can verify that $\sin(x)$ *is the exact solution of our equation. This example illustrates the exactitude of the Adomian decomposition method.*

Example 3.4.2 *Consider the Volterra integral equation*

$$u(x) = \cos(x) + \int_0^x u(t)\cos(x^2)\mathrm{d}t \ .$$

We have that

$$
\begin{aligned}
u_0(x) &= \cos(x) \ , \\
u_1(x) &= \int\limits_0^x \cos(x^2) u_0(t) \mathrm{d}t = \int\limits_0^x \cos(x^2)\cos(t)\mathrm{d}t = \sin(x)\cos(x^2) \ , \quad \textit{and} \\
u_2(x) &= \int\limits_0^x \cos(x^2) u_1(t) \mathrm{d}t = \cos(x^2) \int\limits_0^x \sin(t)\cos(t^2)\mathrm{d}t \ .
\end{aligned}
$$

Note that the function u_2 *can not be evaluated exactly. Therefore the Adomian decomposition method can be used, for instance, to only approximate the solution.*

4. Application and conclusion

4.1 Application

Consider the following Volterra integral equation of the second kind with weakly singular kernel:

$$u(x) = f(x) - \int_0^x K\left(x, s, u(s)\right) \mathrm{d}s$$

and u_n , $n \in \mathbb{N}$ the term of the Adomian decomposition series. For easy implementation of the Adomian decomposition method on computer, we need to descritize the term u_n of the decomposition series. By using Euler approximation method we have:

$$u_n(x) = h \sum_{i=0}^{M} K\left(x, ih, u_{n-1}(ih)\right) \ , \quad \text{for any} \quad n \geq 1$$

where M is a positive integer and $h = \dfrac{x}{M}$.
Now for application consider the following Volterra integral equations of the second kind with weakly singularity.

$$y(t) = 1 - \int_0^t (t-s)^{-\frac{1}{2}} y(s) \mathrm{d}s, \quad t \in [0,1] \tag{4.1}$$

with the exact solution

$$y(t) = \exp(\pi t) \times \text{erfc}(\sqrt{\pi t}) \ \ \text{where} \ \ \text{erfc}(t) = \frac{2}{\sqrt{\pi}} \int_t^{\infty} e^{-s^2} \mathrm{d}s = 1 - \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2} \mathrm{d}s \ ,$$

$$y(t) = t^{\frac{1}{2}} + \frac{3}{8}\pi t^2 - \int_0^t (t-s)^{-\frac{1}{2}} y^3(s) \mathrm{d}s, \quad t \in [0,2] \tag{4.2}$$

$$y(t) = t^{\frac{1}{2}} + \frac{16}{15} t^{\frac{5}{2}} - \int_0^t (t-s)^{-\frac{1}{2}} y^4(s) \mathrm{d}s, \quad t \in [0,1] \tag{4.3}$$

$$y(t) = -\sqrt{\pi} \int_0^t (t-s)^{-\frac{1}{2}} \left(y(s) - sin(s)\right)^3 \mathrm{d}s, \quad t \in [0,1] \ . \tag{4.4}$$

The equation 4.1 satisfies the conditions of Theorem 2.3.4 of existence and uniqueness of solution.The equations 4.2 and 4.3 have the exact solution $y(t) = t^{\frac{1}{2}}$, and this solution is also unique [S. 91]. We also assume that the equation 4.4 satisfies the conditions of existence and uniqueness of solution [S. 91].

A computer program for solving the equations was written and is shown in Appendix A. For equation 4.1 a computer simulation was obtained for $t \in [0,1]$ and the table below shows the approximated values and the error for $t \in [0,1]$ with the step size of 0.2.

Variable t_i	Exact value	Numerical value $y(t_i)$	The error
0.0	1.0	1	0.0
0.2	0.491648482055	0.512278059977	-0.020629577922
0.4	0.396652783653	0.408696794947	-0.0120440112934
0.6	0.343676543316	0.349833274891	-0.00615673157512
0.8	0.308158031935	0.314297283445	-0.00613925150977
1.0	0.282059176176	0.286831705842	-0.00477252966639

For the equation 4.4 we also obtained a computer simulation for $t \in \mathbb{N}$ with a step size of 0.1. The result of the simulation is tabulated in the following table:

The variable t_i	Numerical value $y(t_i)$
0.0	0
0.1	0.000197918791057
0.2	0.00648495089771
0.3	0.035638458799
0.4	0.106911265033
0.5	0.232010751402
0.6	0.410510188562
0.7	0.633702178796
0.8	0.892896515075
0.9	1.18610255318
1.0	1.51928041682

A computer simulation was also obtained for the equations 4.2 and 4.3 for $t \in [0,2]$ and $t \in [0,3]$, with a step size of 0.2 respectively and the result is shown in the following tables 3 and 4 for the respective equations.

	Table 3		
Variable t_i	Exact value	Numerical value $y(t_i)$	The error
0.0	0.0	0.0	0.0
0.2	0.4472135955	0.446998540595	0.000215054905074
0.4	0.632455532034	0.529713861945	0.102741670089
0.6	0.774596669241	0.482268057023	0.292328612219
0.8	0.894427191	-4.04064837699e+14	4.04064837699e+14
1.0	1.0	-2.34179725811e+48	2.34179725811e+48
1.2	1.09544511501	-1.72224392244e+77	1.72224392244e+77
1.4	1.18321595662	-7.15277251639e+104	7.15277251639e+104
1.6	1.26491106407	-2.31614316344e+129	2.31614316344e+129
1.8	1.3416407865	-4.86838133505e+151	4.86838133505e+151
2.0	1.41421356237	-1.81489801941e+172	1.81489801941e+172

	Table 4		
Variable t_i	Exact value	Numerical value $y(t_i)$	The error
0.0	0.0	0.0	0.0
0.2	0.4472135955	0.458787004827	-0.011573409327
0.4	0.632455532034	0.655721784408	-0.0232662523744
0.6	0.774596669241	0.625125740221	0.14947092902
0.8	0.894427191	-1.2096251878	2.1040523788
1.0	1.0	-5.74037094983e+14	5.74037094983e+14
1.2	1.09544511501	-1.7437416424e+51	1.7437416424e+51
1.4	1.18321595662	-2.14495684602e+85	2.14495684602e+85
1.6	1.26491106407	-1.38998026621e+117	1.38998026621e+117
1.8	1.3416407865	-6.00289688102e+146	6.00289688102e+146
2.0	1.41421356237	-2.38619181704e+174	2.38619181704e+174
2.2	1.48323969742	-1.2760442776e+200	1.2760442776e+200
2.4	1.54919333848	-1.36039042692e+224	1.36039042692e+224
2.6	1.61245154966	-4.22341642026e+246	4.22341642026e+246
2.8	1.67332005307	-5.41930052264e+267	5.41930052264e+267

4.2 Conclusion

We have discussed the Adomian decomposition method and have applied its concept to solve the Volterra integral equations. After applying it to the Volterra integral equations with a non-singular kernel, it was noted that the results can converge to exact solution. For the non-linear Volterra integral equations with weakly singular kernels the method gives the opposite results, which mean that the results diverges away from the exact solution.

Comparing the results obtained in this method with those from the rational basis method [S. 91], the method we discussed had larger values of errors, so we could say that it fails and the rational basis method is a more improved way of solving Volterra integral equations with weakly singular kernels.

Appendix A.

A.1 Program

Python grogram implemented to solve the Volterra integral equations, in chapter 4.1 , by using the Adomian Decomposition method.

```
from __future__ import division
from scipy import *
from scipy.integrate import quad
from math import *
import Gnuplot

N=20
M=5
XMAX=2
def erfc(z):
    L=2/sqrt(pi)
    a=quad(lambda t: exp(-t**2) ,0. ,z)
    value = 1-a[0]*L
    return value

#this function calulate the exact solution of the first equation
def exact(t):
    y = exp(pi*t)*erfc(sqrt(pi*t))
    return y

# the function f in different cases correspond to the first term u_{0}
##def f(t):
##   return sqrt(t)+(3.0/8.0)*pi*t**2
def f(t):
    return 0
##def f(t):
    return 1
##def f(t):
##    return sqrt(t)+16/15*t**(5/2)

## g and h define the kernel
def g(t,x):
    return 1/sqrt(t-x)

def h(t):
    return sqrt(pi)*(t)**3
```

```
# range of values of u_{0}(t)
def initialize(N):
    Ydepart=range(0,XMAX*N)
    for i in range(0,XMAX*N):
        Ydepart[i]=f((1/N)*i)
    return Ydepart

# range of values of u_{n}(t)
def integral(t,Yprec,N):
    valeur=0
    for i in range(0,t*N):
        valeur=valeur+g(t,i/N)*h(Yprec[i]-sin(i/N))*1/N
    return -1*valeur

def calcul(M,N):
    Y=range(0,M)
    Y[0]=initialize(N)
    for j in range(1,M):
        Y[j]=range(0,XMAX*N)
        for i in range(0,XMAX*N):
            Y[j][i]=integral(i/N,Y[j-1],N)
    return Y

# range of values of u(t)
def sommation(Y,M,N):
    Final=range(0,XMAX*N)
    for i in range(0,XMAX*N):
        valeur=0
        for j in range(0,M):
            valeur=valeur+Y[j][i]
        Final[i]=valeur
    return Final

Y=sommation(calcul(M,N),M,N)
Z= range(0,XMAX*N)
pas = int(0.1*N)
for i in arange (0,XMAX*N,pas):
    #Z[i]= Y[i]
    print i/N,'\t',  Y[i]
```

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