Analogue Gravity

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Abstract

Analogue models of gravity have a long and distinguished history, dating back to the earliest days of general relativity. In this essay we outline some of this rich background paying attention to more modern applications of analogue models to quantum condensed matter systems as exemplified in a Bose-Einstein condensate. We conclude the essay with a discussion of the emergence of cosmological models in analogue systems.

اَلخلَاصةُ:

مُنذ انّ بدَأت ثوُرة النِسبية العَامة لِلبيرت انشتَين ظلّ العَلمَاء يُفكرُون في نموذج معَملِي لَاختِبَار النَظرية وَ مَدي صحتِهَا . عَليه هذَا وَاحد مِن نمَاذج كثيِرة لدرَسة النظرية ، في هذَا المشروعَ نود انّ نستعَمل مكنِيكَا الكم (بوزو اَنشتين كوندنسيشن) اَو المَادة الكمية المُكثفة لدرَاسة الجَاذبية في المعَمل وَ ايضَا سنختِم المشرُوعَ بمُنَاقَشة النموذج الكوني بَاستعَمَال هذهِ المَادة.

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1. Introduction to Fluid Mechanics

1.1 Introduction

This essay highlights and reviews the concept of analogue gravity in general relativity. Before we start our discussion let us define the word $analogue$ as the simulation of theoretical ideas; in other words, how theoretical results are translated into experimental science. Analogies play very important roles in physics and mathematics, and can provide new ways of looking at problems. For instance, analogue gravity is more theoretical than experimental, as it includes abstract mathematics, and forms a connection between topics in fluid mechanics and phenomena in general relativity. It might provide the key to understanding more about what gravity is and how it works.

A very well-known such analogy uses sound waves in a moving fluid as an analogue for light waves in curved spacetime. Supersonic flows can generate a dumb hole, an acoustic analogue of the black hole in general relativity, and we can extend this idea to calculate the Hawking radiation from the acoustic horizon. But the most important use of the analogue model is how to test the quantum field theory over curved spacetime.

Analogue models of gravity relate the motion of disturbances in a fluid to the dynamics of a scalar field in a spacetime quantitatively endowed with an effective "acoustic metric", whose components depend on fluid variables such as density, sound speed and velocity. A good example of emergent gravity, in this essay in particular, is the analogue model based on fluid mechanics or the fluid dynamics approximation to Bose Einstein Condensation (a quantum fluid). We shall thus focus our attention on the analogue gravity system established by the propagation of linearised perturbations in a Bose-Einstein condensate. Since this model has useful properties, we can use it as stepping stone to the case of quantum gravity.

In fact some of the applications of these analogue model are interesting for experimental reasons. As we have mentioned, others are useful for the new light that they shed on difficult theoretical questions. However the acoustic metric is not unique and analogue gravity is not a full description of general relativistic gravity.

We start our discussion with some concepts from fluid mechanics which can pave the way for a reader a new to this area. We review concepts such as velocity, stress, and pressure. These play an important role in fluid flow. As we know, all fluid motion is governed by conservation equations. We will try to use mathematical tools to derive the equation of motion of sound waves, in preparation for the derivation the equation of acoustic waves in four dimensions (Lorentz geometry). We do this in the hope that these ideas about fluid can assist us in becoming more familiar with analogue gravity.

In the next chapter we will start to discuss the main idea of this essay, starting with the background of the analogue model, and why we are interested in this model. After that we shall try to connect some topics in general relativity to fluid mechanics. By the end of this chapter we will have derived an acoustic metric of the disturbance field in the Lagrangian formulation.

In the last chapter we use the analogue model for Bose-Einstein condensation. The main idea is to test quantum field theory over curved spacetime. We can also work with cosmological geometry (Friedmann universe). With these examples we will be able to complete an analogue model for gravity, hoping that it will help us work in quantum gravity.

1.2 Concepts of Fluid Mechanics

In order to describe fluid flow, we need to be able to deal with characteristic fluid properties which are different at different spatial positions and times. Mathematically we model this situation with variables that describe the physical state of a fluid usually as function of time and spatial postion. As such, the mathematical model of fluid dynamics is based on the continuum hypothesis. Which describes fluid variables with differentiable fields.

Mass Density

The scalar-valued function $\rho = \rho(\underline{x}, t)$ describes the mass density in a given fluid at any time t at any position $\underline{x}$. It can be defined as

$$\rho(\underline{x}, t) = \lim_{\Delta V \longrightarrow 0} \frac{\Delta m}{\Delta V}, \tag{1.1}$$

where $\triangle m$ is the mass of small volume of the fluid.

Flow Velocity

The vector-valued function

$$\underline{v} = \underline{v}(\underline{x}, t) \tag{1.2}$$

gives the velocity of a fluid element at time t and position $\underline{x}$.

Body Forces

A body force is an external force that acts on the volume of a body. The unit of a body force are force per units volume; compare this to pressure (a surface force) which has units of force per unit area. Formally, the body force is defined as:

$$\underline{f} = \lim_{\Delta V \longrightarrow 0} \frac{\Delta \underline{F}}{\Delta V}.$$

The most common example of a body forces is the acceleration due to gravity, $\underline{g}$.

Vorticity And Velocity Potential

We can define the vorticity of a fluid flow as the curl of the velocity:

$$\underline{\omega} = \nabla \times \underline{v}$$

The vorticity at a point is a measure of the local rotation, or spin, of a fluid element at that point. If $\underline{\omega} = \underline{0}$ the fluid said to be *irrotational*, and we can write the velocity in term of a "velocity potential" ϕ as

$$\mathbf{v} = -\nabla\phi \ ,$$

Stress Tensor

Stress is the internal distribution of force per unit area that balances and reacts to external loads applied to a body. It is a second-order tensor with nine components, but can be fully described with six components due to symmetry in the absence of body moments. Stress is often broken down into its shear and normal components as these have unique physical significance.

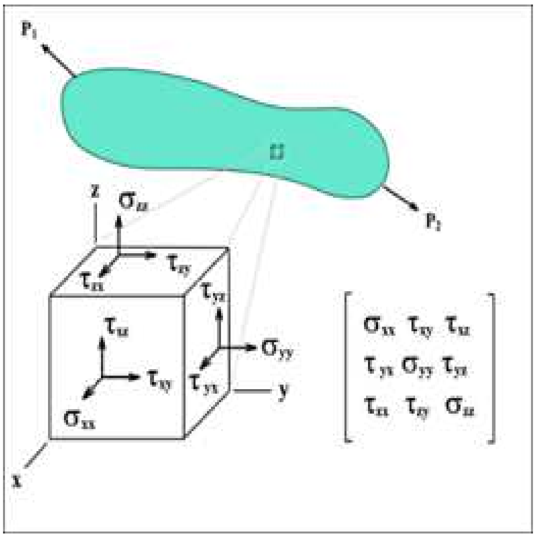

Figure 1.1: This picture shows the distribution of force, and the corresponding stress tensor.

The stress tensor can be defined by

$$dF_i = \sum_{j=1}^{N} \sigma_{ij}\, dA_j,$$

where the dF_i are the components of the resultant force vector acting on a small area dA which can be represented by a vector dA_j perpendicular to the area element (see Fig 1.1), facing outwards and with length equal to the area of the element. σ_{ij} is a symmetric tensor taking the form

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\partial_i v_j + \partial_j v_i\right).$$

σ_{ij} is the stress tensor naturally decompose into an isotropic trace part

$$\sigma^p_{ij} = -p\delta_{ij}$$

and a traceless part

$$\sigma^\mu_{ij} = \mu\left(\partial_i v_j + \partial_j v_i\right).$$

where p is isotropic pressure and μ is the "viscosity" which characterise friction in the fluid.

1.3 Rate of Change Following the Fluid

Consider a function $f(\underline{x}, t)$ (see Fig 1.2) as representing some quantity of interest in the fluid. The rate of change of f as viewed by an observer moving with the fluid is

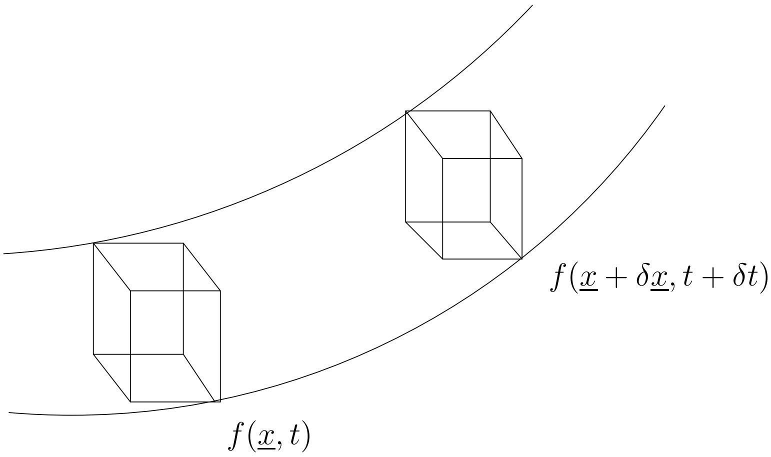

Figure 1.2: This picture shows the the change of the quantity f along stream line.

$$\delta f = f(\underline{x} + \delta\underline{x}, t + \delta t) - f(\underline{x}, t)$$

where $\delta\underline{x} = \underline{v}\delta t$ because we are following a streamline.

$$\delta f = \frac{\partial f}{\partial t}\delta t + \delta\underline{x}.\nabla f$$

$$\Rightarrow \delta f = (\frac{\partial f}{\partial t} + \delta\underline{v}.\nabla f)\delta t. \quad (1.3)$$

By dividing equation (1.3) by δt, we can define the *convective* derivative as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v}.\nabla f. \quad (1.4)$$

This represents the rate of change of f as measured by an observer moving with fluid.

1.4 Conservation of Mass

Consider an arbitrary closed volume V with surface S containing fluid characterised by mass density ρ, velocity vector $\underline{v}$, and pressure p. The principle of conservation of mass asserts that the rate of change of mass of the fluid inside the volume V equals the rate at which mass flows into the volume V through S.

Let us take a positive surface element area $d\underline{S}$, with unit normal on S pointing out of the volume V, so the volume of the fluid per unit time leaving through $d\underline{S}$ is $\underline{v}.d\underline{S}$, and the mass of the fluid per unit time leaving the volume through $d\underline{S}$ is $\rho\underline{v}.d\underline{S}$. The mass of the fluid in a small volume element dV is ρdV, thus the equation of mass conservation is

$$\frac{d}{dt}\int_V \rho dV = -\int_S \rho\underline{v}.d\underline{S}. \tag{1.5}$$

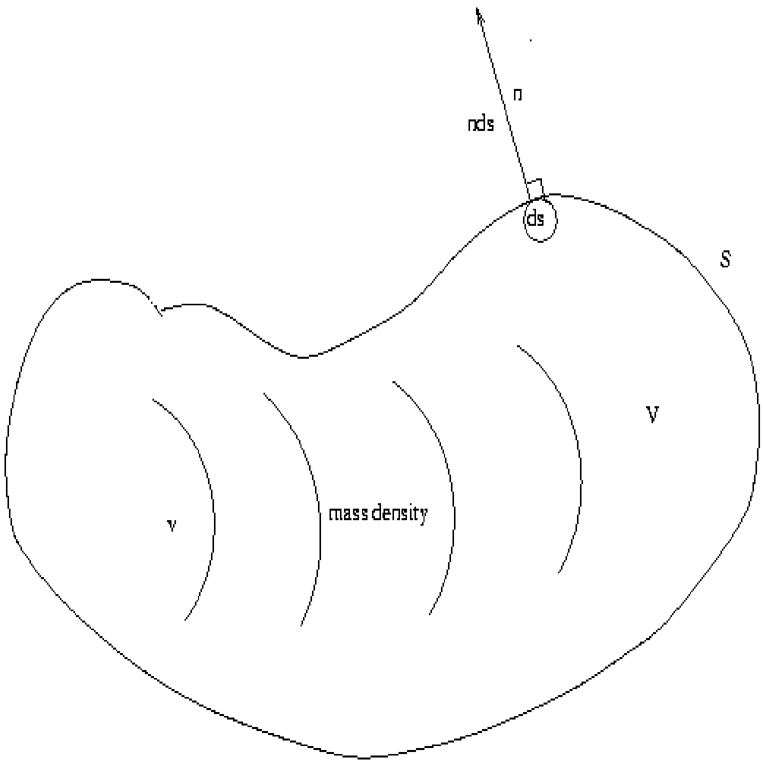

Figure 1.3: Fluid of density ρ and velocity $\underline{v}$ in a volume V, fixed in space, bounded by a fixed surface S. At the element of the surface area $d\underline{S}$, the outward normal is $\underline{n}$.

If $\underline{v}$ and ρ are continuously differentiable functions of $\underline{x}$ and t then the surface integral on RHS of this equation can replaced, using the divergence theorem, by

$$\frac{d}{dt}\int_V \rho dV = -\int_V \nabla.(\rho\underline{v})$$

$$\Rightarrow \int \left[\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{v}) \right] dV = 0, \tag{1.6}$$

and since the volume was chosen to be arbitrary, we get

$$\frac{\partial \rho}{\partial t} + \nabla. (\rho \underline{v}) = 0. \tag{1.7}$$

This is the equation of conservation of mass.

1.5 Conservation of Momentum

As in Section 1.1, we consider a volume V fixed in space, bounded by a fixed surface S. Let the body force per unit volume (for example the acceleration due to gravity) be $\rho \underline{f}(\underline{x}, t)$. The total body force is therefore

$$\underline{F}_B = \int \rho \underline{f}(x, t) dV,$$

and the stress on the surface in the direction j is

$$(F_s)_i = \int \sigma_{ij} dS_j.$$

The total momentum inside the volume is

$$\underline{P} = \int \rho \underline{v} dV,$$

and the flux density through the surface S is

$$\underline{\Phi} = \int \rho \underline{v}\ \underline{v}.d\underline{S}.$$

The equation of conservation of momentum in integral form asserts that the rate of change of momentum of the fluid in V, plus the rate at which momentum leaves V through S, is equal to the total force exerted on the fluid:

$$\frac{d}{dt}\underline{P} = -\underline{\Phi} + \underline{F}_B + \underline{F}_s.$$

$$\Rightarrow \frac{d}{dt} \int_V \rho \underline{v} dV + \int_S \rho \underline{v}(\underline{v}.d\underline{S}) = \int_V \rho \underline{f} dV + \int_S \sigma d\underline{S}. \tag{1.8}$$

Applying the divergence theorem to the above integral we get

$$\int_V \frac{\partial(\rho v_i)}{dt} dV + \int_V \frac{\partial(\rho v_i v_j)}{\partial x_j} dV = \int_V f_i dV + \int_V \frac{\partial(\sigma_{ij})}{\partial x_j} dV,$$

and, since the volume V was chosen to be arbitrary, we obtain

$$\frac{\partial(\rho v_i)}{\partial t} + \frac{\partial(\rho v_i v_j)}{\partial x_j} = \rho f_i + \frac{\partial(\sigma_{ij})}{\partial x_j}. \quad (1.9)$$

This is the differential version of the law of conservation of momentum. [Cha00].

2. Analogue Model

2.1 Background

The basic physics is simple: consider a moving fluid, through which we pass sound waves. The fluid will drag the sound waves along the stream lines as shown in Fig 2.1. As we shall see, if the speed of the fluid anywhere becomes supersonic, the sound waves will never be able to make their way back upstream. This phenomenon implies the existence of a"dumb hole" (the analogue of a black hole from general relativity); we can define a dumb hole as the region from which sound waves cannot escape, and the behaviour of sound waves is very similar to light waves in general relativity (i.e. when the light waves approach the event horizon of a black hole).

Our next step is build a physical and mathematical framework connecting the physics of acoustics in fluid flow and some significant features of general relativity. In this sense we can claim to have an analogue model of some aspect of gravity.

Now, the feature of general relativity that one typically captures in analogue models is the kinematics that have to do with how fields (classical or quantum) behave in curved spacetime. The essence of the analogue model is the existence of an effective metric that captures the idea of curved spacetime which arises naturally in general relativity.

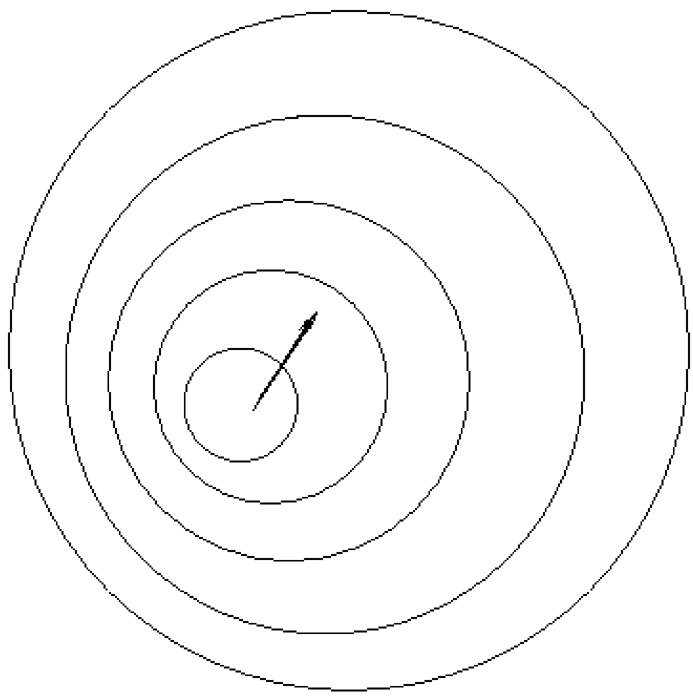

Figure 2.1: A moving fluid will drag sound pulses along with it.

The idea above can be converted into quantitative mathematical statements and it is for this reason that analogue models are of physical interest. There are two different ways to tackle this problem: geometrical acoustics and physical acoustics.

2.2 Geometrical Acoustics

Assume that we have a fluid flow and we pass sound waves through it. We denote the speed of sound relative to the fluid by c. Let the velocity of the fluid be $\underline{v}$, relative to the laboratory. The propagation of the sound waves with respect to the fluid along the direction determined by the unit vector $\underline{n}$ is then governed by the equation

$$\frac{d\underline{x}}{dt} = c\underline{n} + \underline{v}. \tag{2.1}$$

We can rearrange the equation, and multiply both sides by dt, to get a quadratic equation which is similar to intervals in general relativity. We then have that

$$-c^2 dt + (d\underline{x} - \underline{v}dt)^2 = 0 \tag{2.2}$$

$$\Rightarrow -(c^2 - \underline{v}^2)dt^2 - 2\underline{v}.d\underline{x}dt + d\underline{x}^2 = 0. \tag{2.3}$$

This equation defines a double cone associated with each point in space and time. This is associated with a conformal class of Lorentzian metric. Also equation (2.3) looks like an interval in general relativity.

We call the metric associated with this interval the "acoustic metric". In this case it is given by

$$g_{\mu\nu} = \Omega^2 \begin{pmatrix} -(c^2 - \underline{v}^2) & \vdots & -\underline{v} \\ \cdots\cdots\cdots\cdots & \cdots & \cdots \\ -\underline{v} & \vdots & I \end{pmatrix}$$

where Ω is an unspecified, but non-vanishing function. We have seen in geometrical acoustics that the derivation of the mathematics is easy, and we can deduce the causal structure of the spacetime. We cannot obtain a unique effective metric, but only find it up to conformal transformations. Therefore we will finish with geometrical acoustics, and turn to physical acoustics which has the advantage of giving us the specific effective metric, and accommodates the wave equation for the sound waves [BLV06]

2.3 Physical Acoustics

A disturbance in a homogeneous, static, irrotational, and inviscid fluid flow has a velocity potential $\delta\phi$ which satisfies the well-known wave equation

$$\partial_t^2 \delta\phi = c^2 \nabla^2 \delta\phi \tag{2.4}$$

where c is the speed of sound.

We would like to generalise this result to fluids that are non-homogeneous, or to fluids that are in motion, possibly even in non-steady motion. We derive the wave equation for this case.

Theorem: If a fluid is barotropic, (i.e. the pressure is a function of the density only), inviscid, and the flow is irrotational, then the equation of motion for the velocity potential $\delta\phi(\underline{x}, t)$ describing the acoustic disturbances is identical to the d'Alembertian equation of motion for a minimally-coupled massless scalar field propagating in a four-dimension Lorenzian geometry

$$\Delta\delta\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\delta\phi) = 0 \tag{2.5}$$

with "acoustic metric"

$$f^{\mu\nu} = \frac{\rho_0}{c^2}\begin{pmatrix} -1 & \vdots & -v_0^j \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & -c^2\delta^{ij} - v_0^i v_0^j \end{pmatrix}.$$

. This metric describes a four-dimensional Lorentzian geometry. The metric depends on the density, velocity of flow, and local speed of sound in the fluid.

Proof.

Suppose that we have a fluid flow. The fundamental equations of the dynamics of the fluid are the equations of continuity:

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\underline{v}) = 0, \tag{2.6}$$

and Euler's equation (equivalent to $F = ma$ applied to a small element of fluid)

$$\frac{\partial}{\partial t}\rho v_i + \frac{\partial}{\partial x_j}(\rho v_i v_j) = F_i, \tag{2.7}$$

where

$$F_i = \rho f_i + \partial_j\sigma_{ij}$$

represents the total force on a fluid element. It combining the body force and stress tensor. We can write the LHS of equation (2.7) as

$$\frac{\partial\rho}{\partial t}v_i + \rho\frac{\partial v_i}{\partial t} + \rho v_j\partial_j v_i + v_j\partial_j(\rho v_i) = \rho\frac{D}{Dt}v_i + v_j\left(\frac{\partial\rho}{\partial t} + \partial_j(\rho v_i)\right).$$

From equation (2.6) the second term on the RHS vanishes, giving us

$$\rho\frac{D\underline{v}}{Dt} = \underline{F}. \tag{2.8}$$

Assume the fluid is inviscid (zero viscosity), with the only force present being those due to the pressure. Using the identity

$$(\underline{v}.\nabla)\underline{v} = \underline{v} \times \nabla \times \underline{v} - \triangledown\left(\frac{1}{2}v^2\right), \tag{2.9}$$

Euler's equation can be written as

$$-\frac{1}{\rho}\nabla p = (\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\nabla)\underline{v}) = \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \times \nabla \times \underline{v} - \nabla(\frac{1}{2}v^2)\right). \tag{2.10}$$

The second term on the RHS of equation (2.9) equals zero since the fluid is irrotational (and we can locally introduce the velocity potential $\underline{v} = \nabla\phi$); Euler's equation now reduces to

$$\frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla\left(\frac{1}{2}v^2\right). \tag{2.11}$$

This is a version of Bernoulli's equation. We will linearise ρ,p and ϕ around their background values $\rho_0(\underline{x}, t)$,$p_0(\underline{x}, t)$ and $\phi_0(\underline{x}, t)$. Thus

$$p = p_0 + \varepsilon p_1 + O(\varepsilon^2), \tag{2.12}$$

$$\rho = \rho_0 + \varepsilon \rho_1 + O(\varepsilon^2), \tag{2.13}$$

and

$$\phi = \phi_0 + \varepsilon \phi_1 + O(\varepsilon^2); \tag{2.14}$$

from now on, we shall work only to first order in ε . Substituting this into equation (2.6) gives

$$-\frac{\partial \rho_0}{\partial t} + \nabla(\rho_0 \underline{v}_0) = 0, \tag{2.15}$$

$$-\frac{\partial \rho_1}{\partial t} + \nabla(\rho_1 \underline{v}_0 + \rho_0 \underline{v}_1) = 0. \tag{2.16}$$

If we take the fluid to be barotropic (i.e. p is function of ρ only), we can define the specific enthalpy $h(p)$ as a function of p.

$$h(p) = \int \frac{dp}{\rho(p)} \Rightarrow \nabla h = \frac{1}{\rho}\nabla p. \tag{2.17}$$

The barotropic condition implies that

$$h(p) = h\left(p_0 + \varepsilon p_1 + O(\varepsilon^2\right) = p_0 + \varepsilon\frac{p_1}{\rho_0} + O(\varepsilon^2). \tag{2.18}$$

Neglecting the higher-order terms, and using the linearised Euler equation, we have that

$$-\frac{\partial \phi_0}{\partial t} + h_0 + \frac{1}{2}(\nabla \phi_0)^2 = 0 \tag{2.19}$$

$$-\frac{\partial \phi_1}{\partial t} + \frac{p_1}{\rho_0} - v_0.\nabla \phi_1 = 0. \tag{2.20}$$

We can rearrange the last equation to find p_1:

$$p_1 = \rho_0 \left(\frac{\partial \phi_1}{\partial t} + \underline{v}_0.\nabla \phi_1 \right). \tag{2.21}$$

By using the barotropic assumption, the mass density is given by

$$\rho_1 = \frac{\partial \phi}{\partial p} p_1 = \frac{\partial \rho}{\partial p} \rho_0 \left(\frac{\partial \phi_1}{\partial t} + \underline{v}_0.\nabla \phi_1 \right). \tag{2.22}$$

Substituting equations (2.21) and (2.22) into the linearised equation of continuity, we find that the wave equation of the fluid flow in four dimensions is

$$-\partial_t \left[\frac{\partial \rho}{\partial p} \rho_0 \left(\frac{\partial \phi_1}{\partial t} + \underline{v}_0.\nabla \phi_1 \right) \right] + \nabla . \left[\rho_0 \nabla \phi_1 - \frac{\partial \rho}{\partial p} \rho_0 \left(\frac{\partial \phi_1}{\partial t} + \underline{v}_0.\nabla \phi_1 \right) \right] = 0. \tag{2.23}$$

This equation describes the propagation of the linearised scalar potential ϕ_1. Therefore we can describe the propagation of acoustic disturbances. Note the time-dependent and position-dependent coefficients in this wave equation.

The local speed of sound is

$$c^{-2} = \frac{\partial \rho}{\partial p}. \tag{2.24}$$

We can rewrite equation (2.23) using 4×4 matrix $f^{\mu\nu}$ defined by

$$f^{\mu\nu} = \frac{\rho_0}{c^2} \begin{pmatrix} -1 & \vdots & -v_0^j \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & -c^2\delta^{ij} - v_0^i v_0^j \end{pmatrix}. \tag{2.25}$$

Here, μ, ν are indices that run over $1, 2, 3, 4$. We then use Einstein's convention to write equation (2.23) as

$$\partial_\mu (f^{\mu\nu} \partial_\nu \phi_1) = 0. \tag{2.26}$$

The remaining steps are applications of techniques of curved-space four-dimensional Lorentzian geometry. By comparing equation (2.26) to the definition

$$\triangle\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\mu\phi) \tag{2.27}$$

of the d'Alembratian operator for a scalar field ϕ moving in a metric $g_{\mu\nu}$, we see that equation (2.26) corresponds to acoustic disturbances moving in effective "acoustic metric" $g_{\mu\nu}$ satisfying

$$f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}. \tag{2.28}$$

Taking the determinant of both sides, we see that

$$\frac{1}{f} = g^2\frac{1}{g} \Leftrightarrow g = \frac{1}{f};$$

therefore the acoustic metric is given in term of $f_{\mu\nu}$ by

$$g_{\mu\nu} = \frac{1}{\sqrt{-f}}f_{\mu\nu}.$$

Expanding the determinant yields

$$\det(f^{\mu\nu}) = \frac{1}{f} = \left(\frac{\rho_0}{c^2}\right)^4\left[(-1)(c^2 - v_0^2) - (v_0)^2\right].c^4 = -\frac{\rho_0^4}{c^2}. \tag{2.29}$$

Thus

$$\sqrt{-g} = \frac{\rho_0^2}{c}.$$

The inverse of the acoustic metric is therefore

$$g^{\mu\nu}(t,\underline{x}) = \frac{1}{\rho_0 c}\begin{pmatrix} -1 & \vdots & -v_0^j \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ -v_0^i & \vdots & -c^2\delta^{ij} - v_0^i v_0^j \end{pmatrix}, \tag{2.30}$$

giving us that

$$g_{\mu\nu}(t,\underline{x}) = \frac{\rho_0}{c}\begin{pmatrix} -(c^2 - v_0^2) & \vdots & -v_0^j \\ \cdots\cdots\cdots & \cdots & \cdots \\ -v_0^i & \vdots & \delta^{ij} \end{pmatrix}. \tag{2.31}$$

Therefore, the acoustic interval in four dimensions is given by.

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{\rho_0}{c}\left[-c^2dt^2 + (dx^i - v_0^i dt)\delta^{ij}(dx^j - v_0^j dt)\right]. \tag{2.32}$$

So this last equation above completes our proof of the theorem, and we can generalise this further in the analogue model. ■

Note that:

- The signature of this effective metric is indeed $(-,+,+,+)$ as it should be to be regarded as Lorentzian.

- In physical acoustics, it is the inverse metric density,

$$f^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$$

. Which is of fundamental significance when it comes to deriving the wave equation rather than the metric $g_{\mu\nu}$ itself.

- The acoustic metric is a smooth function and it can be defined everywhere.

- Our equation (2.26) for propagation of the acoustic disturbance is covariant in the general relativity sense.

2.4 Simple Lagrangian

Let us first define the principle of the least action. This is a variational principle which, when applied to the action of a mechanical system, can be used to obtain the equations of motion for that system. The principle leads to the development of the Lagrangian and Hamiltonian formulations of classical mechanics. The principle remains central in modern physics and mathematics, and has applications in the theory of relativity, quantum mechanics and quantum field theory.

In this section particularly we want use the Lagrangian approach to generalise the idea of the acoustic metric. Assume we have a four-dimensional containing a single scalar field ϕ. The action, denoted by $\mathcal{S}[\phi]$, is given by

$$S[\phi] = \int d^4x\mathcal{L}(\partial_\mu\phi, \phi). \tag{2.33}$$

where $\mathcal{L}(\partial_\mu\phi, \phi)$ depends on the the field ϕ and it is derivative $\partial_\mu\phi$ and is called the "Lagrangian density". The principle of the least action implies that the action should be stationary when the field satisfies its equation of motion:

$$\delta S[\phi] = \delta \int d^4x\mathcal{L}(\partial_\mu\phi, \phi) = 0. \tag{2.34}$$

We expand the field about its background value:

$$\phi(t, x) = \phi_0(t, x) + \epsilon\phi_1(t, x) + \frac{\epsilon^2}{2}\phi_2(t, x) + O(\epsilon^3) \tag{2.35}$$

Therefore, the general Lagrangian is

$$\mathcal{L}(\partial_\mu\phi, \phi) = \mathcal{L}(\partial_\mu\phi_0, \phi_0) + \epsilon\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}.\partial_\mu\phi_1 + \frac{\partial\mathcal{L}}{\partial\phi}\phi\right] + \frac{\epsilon}{2}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\phi_2 + \frac{\partial\mathcal{L}}{\partial\phi}\phi_2\right] + \frac{\epsilon^2}{2}\left[\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\partial(\partial_\nu\phi)}\partial_\mu\phi_1\partial_\nu\phi_1 + 2\frac{\partial^2\mathcal{L}}{\partial(\partial_\mu\phi)\partial\phi}.\partial_\mu\phi_1\phi_1 + \frac{\partial^2\mathcal{L}}{\partial\phi^2}\phi_1^2\right] + O(\epsilon^3) \tag{2.36}$$

Substituting this back into the general formula for the action and integrating the second term by parts, we note that the first and second terms give Euler-Lagrange equation for the background field,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{2.37}$$

By using the second variation principle (i.e. second perturbation term), the general Lagrangian of the action can be represented by

$$\frac{\epsilon^2}{2} \int d^4x \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right) \partial_\mu \phi_1 \partial_\nu \phi_1 + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi^2} - \partial_\mu \left[\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial \phi} \right] + O(\epsilon^3)\phi_1 \phi_1 \right) = 0.$$

The parameter ϵ is small, hence we can neglect the higher-order terms, and the equation of motion of the linearised fluctuations is

$$\partial_\mu \left[\left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi}) \right) \partial_\nu \phi_1 \right] - \left[\left(\frac{\partial^2 \mathcal{L}}{\partial \phi^2} \right) - \partial_\mu \left(\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial \phi} \right) \right] \phi_1 = 0. \tag{2.38}$$

This is a second-order differential equation with position dependent coefficients. The effective inverse space-time metric is

$$\sqrt{-g} g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left. \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right|_{\phi_0}. \tag{2.39}$$

Therefore,

$$g^{\mu\nu}(\phi_0) = \left. \left(-\det \left[\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right] \right)^{-1/2} \right|_{\phi 0} \left(\left. \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right|_{\phi_0} \right). \tag{2.40}$$

Therefore, the metric is given by

$$g_{\mu\nu}(\phi_0) = \left. \left(-\det \left[\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right] \right)^{1/2} \right|_{\phi 0} \left(\left. \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\partial(\partial_\nu \phi)} \right|_{\phi_0} \right)^{-1}. \tag{2.41}$$

This is an effective acoustic metric in the Lagrangian density form.

2.5 Acoustic ergospheres and event horizons

In general relativity, we can identify two surfaces in the space time surrounding a rotating black hole (see Fig. 2.2). The *event horizon* is the surface at which the light cones tip over so that light rays cannot escape. The *ergosphere* is the boundary of the region in which particles in which particles are forced to rotate with the black hole.

We can easily define acoustic analogue of these. The *acoustic event horizon* is the surface at which the acoustic light cones tip over, i.e. the radial fluid velocity is equal to the sound speed. This means that sound waves cannot penetrate this surface, giving rise to the acoustic analogue of a black hole, the *dumb hole*. The *acoustic ergosphere* is the surface at which the acoustic metric satisfies $g_{tt} = 0$. The region between this and the event horizon is called the *ergoregion*.

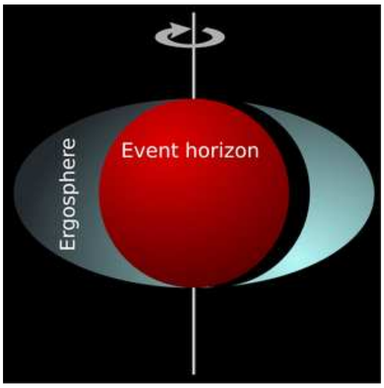

Figure 2.2: This picture show the ergosurface, ergosphere, and event horizon for a rotating black hole

2.5.1 Vortex Geometry

Assume we have a fluid flow. We want to model a draining bathtub by a four-dimensional flow with a linear sink along the z-axis. The ergosphere and acoustic event horizon are both critical in considering the draining bathtub fluid flow. Let us start with the simplifying assumption that the background density ρ is a position-independent constant throughout the flow. The equation of continuity then implies for the radial component velocity that we must have

$$v^{\hat{r}} \propto \frac{1}{r}.$$

In the tangential direction, the requirement that the flow be vorticity free implies, via Stokes' theorem, that

$$v^{\hat{t}} \propto \frac{1}{r}.$$

Since the tangential and radial velocities are both inversely proportional to r, we can rewrite the background velocity potential as

$$\phi(r, \theta) = -A \ln(r/a) - B\theta.$$

The velocity potential of the fluid flow is then

$$\mathbf{v} = -\nabla\phi = \frac{(A\hat{r} + B\hat{\theta})}{r}.$$

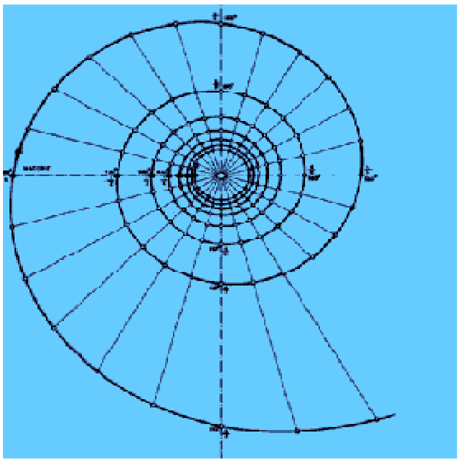

Figure 2.3: This picture shows the ergosphere and event horizon of vortex geometry.

Dropping a position-independent prefactor, the acoustic metric for a draining bathtub is explicitly given by

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r} dt\right)^2 + \left(r d\theta - \frac{B}{r} dt\right)^2 + dz^2.$$

We can factorise this as

$$ds^2 = -\left(c^2 - \frac{A^2 + B^2}{r^2}\right)^2 dt^2 - 2\frac{A}{r} dr dt - 2B d\theta dt + dr^2 + r^2 d\theta^2 + dz^2.$$

A similar metric, restricted to A=0 (no radial flow), and generalised for an anisotropic speed of sound, has been given by Voloik [BLV06]. The Voloik metric is a model for the acoustic geometry surrounding physical vortices in a $He - 3$ superfluid.

In line with previous comments, the vortex fluid flow is seen to possess an acoustic metric that is stably causal and which does not involve closed timelike curves. At large distances it is possible to approximate the vortex geometry by a spinning cosmic string, but this approximation becomes progressively worse as the core is approached. The ergosphere forms where $g_{tt} = 0$, namely at

$$r_{\text{ergosphere}} = \frac{\sqrt{A^2 + B^2}}{c}.$$

Note that the sign of A is irrelevant in defining the ergosphere and ergoregion. The event horizon is where the radial component of the velocity equals the sound speed:

$$r_{\text{horizon}} = \frac{|A|}{c}.$$

The sign of A here distinguishes between two cases. If $A < 0$, the radial velocity is inwards, so sounds waves cannot get *out* (this is called a *future* acoustic horizon). Conversely, if $A > 0$, the radial velocity is outwards; now, sounds waves cannot get in. This is called a *past* acoustic horizon.

3. Quantum Models

3.1 Bose-Einstein Condensates

3.1.1 What Is Bose-Einstein Condensates

A Bose-Einstein condensate (BEC) is a state of matter formed by bosons (i.e. elementary particles having integer spin) cooled to temperatures very near to absolute zero. Under such supercooled conditions, a large fraction of the atoms collapse into the lowest energy quantum state, at which point quantum effects become apparent on a macroscopic scale. The existence of BEC was predicted by Einstein in 1925.

Condensed matter physics is the field of physics that deals with the macroscopic physical properties of matter. In particular, it is concerned with the "condensed" phases that appear whenever the number of constituents in a system is extremely large, and the interactions between the constituents are strong.

We shall show that the propagation of phase perturbations in a BEC can, under certain conditions, closely mimic the dynamics of quantum fields in curved spacetimes. Let us start by deriving the acoustic metric for the BECs system. A Bose gas can be described as a quantum field $\widehat{\Psi}$ satisfying the Schrödinger wave equation:

$$i\hbar\frac{\partial}{\partial t}\widehat{\Psi} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(\underline{x}) + \kappa(a)\widehat{\Psi}^\dagger\widehat{\Psi}\right)\widehat{\Psi}. \tag{3.1}$$

The parameter $\kappa(a)$ describes the interactions between the different bosons in the gas. This quantity can be expressed in terms of the scattering length a as

$$\kappa(a) = \frac{4\pi a\hbar^2}{m}. \tag{3.2}$$

The quantum field $\widehat{\Psi}$ can be separated into a macroscopic (classical) condensate ψ and a quantum fluctuation field $\widehat{\varphi}$: $\widehat{\Psi} = \psi + \widehat{\varphi}$; with $\langle\widehat{\Psi}\rangle = \psi$. Substituting into equation (3.1)

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}(\psi+\widehat{\varphi}) &= \mathcal{L}(\psi+\widehat{\varphi}) + \kappa(a)\left[n_c\psi + 2n_c\widehat{\varphi} + \psi^\dagger\widehat{\varphi}\widehat{\varphi}\right] + \kappa(a)\left[m_c\widehat{\varphi}^\dagger + 2\psi\widehat{\varphi}^\dagger\widehat{\varphi} + \widehat{\varphi}^\dagger\widehat{\varphi}^2\right] \\ &= \mathcal{L}\psi + \mathcal{L}\widehat{\varphi} + \kappa(a)\left[n_c\psi + 2n_c\widehat{\varphi} + \psi^\dagger\widehat{\varphi}\widehat{\varphi} + m_c\widehat{\varphi}^\dagger + 2\psi\widehat{\varphi}^\dagger\widehat{\varphi} + 2\widetilde{n}\widehat{\varphi} + 2\widetilde{m}\widehat{\varphi}\right]. \end{aligned}$$

where

$n_c \equiv |\psi(t,x)|^2$, $m_c \equiv \psi^2(t,x)$, $\widetilde{n} \equiv \langle\widehat{\psi}\psi\rangle$, $\widetilde{m} \equiv \langle\widehat{\psi}\widehat{\psi}\rangle$, $n_T \equiv n_c + \widetilde{n}$, $m_T \equiv m_c + \widetilde{m}$.

Taking expectations, using $\langle\widehat{\varphi}\rangle = 0$, we obtain

$$i\hbar\frac{\partial}{\partial t}\psi = \mathcal{L}\psi + \kappa(a)n_c\psi + \kappa(a)\psi^\dagger\tilde{m} + 2\kappa(a)\psi\tilde{n}.$$

For the fluctuation term we obtain

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\widehat{\varphi} = \mathcal{L}\widehat{\varphi} + \kappa(a)\left[n_c\psi + 2n_c\widehat{\varphi} + \psi^\dagger\widehat{\varphi}^2 + m_c\widehat{\varphi}^\dagger + 2\psi\widehat{\varphi}^\dagger\widehat{\varphi} + 2\tilde{n}\widehat{\varphi} + \tilde{m}\widehat{\varphi}^\dagger\right] - \\ \kappa(a)\left[n_c\psi + \kappa(a)\psi^\dagger\tilde{m} + 2\kappa(a)\psi\tilde{n}\right] \\ = \mathcal{L}\widehat{\varphi} + \kappa(a)\left[\left(2n_c + 2\tilde{n}\right)\widehat{\varphi} + \left(m_c + \tilde{m}\right)\widehat{\varphi}^\dagger\right] + \\ \kappa(a)\left[\psi^\dagger\left(\widehat{\varphi}^2 - \langle\widehat{\varphi}^2\rangle\right) + 2\psi\left(\widehat{\varphi}^\dagger\widehat{\varphi} - \langle\widehat{\varphi}\widehat{\varphi}^\dagger\rangle\right)\right]. \end{aligned}$$

Then by adopting the "self-consistent mean field approximation",

$$\widehat{\varphi}^\dagger\widehat{\varphi}\widehat{\varphi} \simeq 2\langle\widehat{\varphi}^\dagger\widehat{\varphi}\rangle\widehat{\varphi} + \langle\widehat{\varphi}\widehat{\varphi}\rangle\widehat{\varphi}^\dagger \tag{3.3}$$

and neglecting the terms

$$\widehat{\varphi}^2 - \langle\widehat{\varphi}^2\rangle$$

and

$$\widehat{\varphi}^\dagger\widehat{\varphi} - \langle\widehat{\varphi}^\dagger\widehat{\varphi}\rangle,$$

one can arrive at the equation

$$i\hbar\frac{\partial}{\partial t}\psi = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(x) + \kappa(a)n_c\right)\psi + \kappa\left[2\tilde{n}\psi + \tilde{m}\psi^\dagger\right] \tag{3.4}$$

which describe the classical quantum field, and for the fluctuation we have

$$i\hbar\frac{\partial}{\partial t}\widehat{\varphi} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{ext}(x) + 2\kappa(a)n_T\right)\widehat{\varphi} + \kappa(a)\left[m_T\widehat{\varphi}^\dagger\right] \tag{3.5}$$

Here $\tilde{n}$ and $\tilde{m}$ describe the back reaction. The equation for the classical wave function of the condensate is closed only when the back-reaction effect due to the fluctuations is neglected. This called Gross-Pitaeviski approximation. In the general case one will have to solve the equations for the classical field and the fluctuation simultaneously. Adopting the *Madelung representation* for the wave function of the condensate,

$$\psi(t,x) = \sqrt{n_c(t,x)}\exp[-i\theta(t,x)/\hbar] \tag{3.6}$$

and defining an irrotational "velocity field " by $\underline{v} \equiv \nabla\theta$, the equations (3.4) can be rewritten using the Gross-Pitaeviski approximation as a continuity equation plus an Euler equation:

$$\frac{\partial n_c}{\partial t} + \nabla.(n_c\underline{v}) = 0. \tag{3.7}$$

and

$$m\frac{\partial \underline{v}}{\partial t} + \nabla\left(\frac{mv^2}{2} + V_{ext}(t,x) + \kappa(a)n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{3.8}$$

Equations (3.7) and (3.8) are equivalent to those of an irrotational and inviscid fluid. The additional term in equation (3.8) describes the quantum potential

$$V_Q = -\hbar^2\nabla^2\sqrt{n_c}/(2m\sqrt{n_c})$$

and has the dimension of energy. Note that

$$n_c\nabla_i V_Q \equiv n_c\nabla_i\left(-\hbar^2\nabla^2\sqrt{n_c}/(2m\sqrt{n_c})\right) = \nabla_j\left(\frac{-\hbar^2}{2m}n_c\nabla_i\nabla_j\log n_c\right). \tag{3.9}$$

This then justifies the introduction of the so-called "quantum stress tensor"

$$\sigma^Q_{ij} = -\frac{\hbar^2}{4m}n_c\nabla_i\nabla_j\log n_c.$$

This tensor has the dimension of pressure, and may be viewed as an intrinsically quantum anisotropic pressure contributing to the Euler equation. If we write the mass density of the Madelung fluid as $\rho = mn_c$, and use the fact that the flow is irrotational, then the Euler equation takes the form

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + (\underline{v}.\nabla)\underline{v}\right) + \rho\nabla\left(\frac{V_{ext}}{m}\right) + \nabla\left(\frac{\kappa\rho^2}{2m}\right) + \nabla.\sigma_Q = 0. \tag{3.10}$$

The term V_{ext}/m has the dimensions of specific enthalpy, and the term $\frac{\kappa\rho^2}{2m}$ represents the bulk pressure. When the gradient density of the condensate is very small one can neglect the quantum stress tensor. This will lead to the standard hydrodynamic approximation. Because the flow is irrotational, the Euler equation takes the form

$$m\frac{\partial\theta}{\partial t} + \left(\frac{(\nabla\theta)^2}{2m} + V_{ext} + \kappa(a)n_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right) = 0. \tag{3.11}$$

The most important thing is to account for the small quantum perterbations of the system. These quantum perturbations can be described in different ways. Here we are interested in a "quantum acoustic representation"

$$\widehat{\varphi}(t,x) = e^{i\theta/\hbar}\left(\frac{\widehat{n}_1}{2\sqrt{n_c}} - i\frac{\sqrt{n_c}}{\hbar}\widehat{\theta}_1\right) \tag{3.12}$$

where $\widehat{n}_1, \widehat{\theta}_1$, are real quantum fields. Therefore by substituting this representation into the general equation of quantum fluctuations, we obtain a coupled set of equations similar in form to an equation of continuity plus an Euler equation:

$$\partial_t\widehat{n}_1 + \frac{1}{m}\nabla.\left(n_1\nabla\theta + n_c\nabla\widehat{\theta}_1\right) = 0 \tag{3.13}$$

$$\partial_t \widehat{\theta}_1 + \frac{1}{m}\nabla\theta.\nabla\widehat{\theta}_1 + \kappa(a)n_1 - \frac{\hbar^2}{2m}D_2\widehat{n}_1 = 0 \tag{3.14}$$

where D_2 is the second-order operator obtained from linearising the quantum potential,

$$D_2\widehat{n}_1 = \frac{-1}{2}n_c^{-3/2}\left(\nabla^2(n_c^{1/2})\right)\widehat{n}_1 + \frac{1}{2}n_c^{-1/2}\nabla^2(n_c^{-1/2}\widehat{n}_1). \tag{3.15}$$

From the previous equation of the linearised perturbation, it is possible to derive a wave equation for $\widehat{\theta}_1$. We need to substitute the value of $\widehat{n}_1$ determined from the linearisation equation into the equation of continuity. This leads to a partial differential equation that is of second order in time derivatives, but of infinite order in space derivatives. We can construct the 4×4 matrix

$$f^{\mu\nu}(t,x) = \begin{pmatrix} f^{00} & \vdots & f^{0j} \\ \cdots & \cdots & \cdots \\ f^{i0} & \vdots & f^{ij} \end{pmatrix} \tag{3.16}$$

where μ,ν are indices running over $(1,2,3,4)$. The wave equation for θ_1 is then easily rewritten as

$$\partial_\mu\left(f^{\mu\nu}\partial_\nu\widehat{\theta}_1\right) = 0 \tag{3.17}$$

where $f^{\mu\nu}$ is differential operator acting on space only:

$$f^{00} = -\left(\kappa(a) - \frac{\hbar^2}{2m}D_2\right)^{-1},$$

$$f^{0j} = -\left(\kappa(a) - \frac{\hbar^2}{2m}D_2\right)^{-1}\frac{\nabla^j\theta_0}{m},$$

$$f^{i0} = -\frac{\nabla^i\theta_0}{m}\left(\kappa(a) - \frac{\hbar^2}{2m}D_2\right)^{-1},$$

$$f^{ij} = \frac{n_c}{m}\delta^{ij} - \frac{\nabla^i\theta_0}{m}\left(\kappa(a) - \frac{\hbar^2}{2m}D_2\right)^{-1}\frac{\nabla^j\theta_0}{m}.$$

If we perform a decomposition of the field $\widehat{\theta}_1$, we can see that for each wavelength larger than $\hbar/mc_s$, the term coming from linearisation of the quantum potential can be neglected in the previous expressions. In this case the $f^{\mu\nu}$ can by approximated by numbers, instead of differential operators. Then, by identifying

$$\sqrt{-g}g^{\mu\nu} = f^{\mu\nu}, \tag{3.18}$$

the equation for the field $\widehat{\theta}_1$ becomes that of a quantum scalar field over a curved background,

$$\triangle\widehat{\theta}_1 = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\mu\widehat{\theta}_1) = 0. \tag{3.19}$$

By comparing equations (3.18) and (3.19), we can read off the effective acoustic metric for the BEC system,

$$g_{\mu\nu}(t,\underline{x}) = \frac{n_c}{mc_S(a,n_c)} \begin{pmatrix} -\left[c_s(a,n_c)\right]^2 - v^2 & \vdots & -v_j \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots & \cdots \\ -v_i & \vdots & \delta_{ij} \end{pmatrix}. \tag{3.20}$$

This is the effective metric for scalar quantum field theory over curved space-time, in term of the speed of photons $c_S(a,n_c)$, which is

$$c_S(a,n_c) = \frac{\kappa(a)n_c}{m}.$$

The BEC is a good test of the existence of a quantum field over curved space-time.

3.2 Cosmological Geometries

In this example we want to apply techniques of analogue models to the cosmological framework. Our interest here are the Friedman Robertson-Walker [FLRW] geometries. We know that the cosmological particles are produced by the expansion of the universe, so by applying this model, maybe we can understand more about these quantities. To simulate the expansion of the flat universe, we will start by writing the acoustic metric

$$ds^2 = \frac{\rho}{c_s}\left[-(c_s^2 - \underline{v}^2) - 2\underline{v}.d\underline{x}dt + d\underline{x}^2\right]. \tag{3.21}$$

We have two ways to reproduce cosmological spacetimes. Let us take a homogeneous (which claims that the universe looks the same at every point) system, where $\rho(t), c_s(t)$ depend on time. The radial velocity profile of the universe is

$$\underline{v} = (\dot{b}/b)\underline{r},$$

where $b(t)$ is the scalar factor of the universe. We can define a new radial coordinate

$$r_b = r/b$$

In term of which the acoustic metric can be written as

$$ds^2 = \frac{\rho}{c_s}\left[-c_s^2 dt^2 + b^2(dr_b^2 + r_b^2 d\Omega_2^2)\right], \tag{3.22}$$

where Ω_2^2 is solid angle element as observed by an observer on earth. We can then introduce the Hubble constant of the Friedman Roberson-Walker [FLRW] universe as

$$H_b(t) = \frac{\dot{b}(t)}{b(t)}. \tag{3.23}$$

The conservation of momentum and energy implies that

$$\dot{\rho} + 3H_b(t)\rho = 0 \Longrightarrow \rho(t) = \frac{\rho_0}{b^3(t)}. \tag{3.24}$$

This is the equation of continuity. Finally we arrive at the metric of a flat FLRW geometry

$$ds^2 = -T^2(t)dt^2 + a_s^2(dr_b^2 + r_b^2\Omega_2^2) \tag{3.25}$$

where $T(t) = \sqrt{\rho c_s}$ and $a_s = \sqrt{\frac{\rho}{c_s}}b.$. We can find the conformal Friedman time τ , which is related to the laboratory time t, by

$$\tau = \int T(t)dt \tag{3.26}$$

Alternatively, we could choose to start from a fluid at rest $\underline{v} = \underline{0}$ with respect to the laboratory at all times. In this case the acoustic metric is

$$ds^2 = -\rho c_s dt^2 + \frac{\rho}{c_s}d\underline{x}^2. \tag{3.27}$$

Now it is not difficult to imagine a situation in which ρ remains constant, in a sufficiently large region of the space, while the speed of sound decreases with time. This method also reproduces an expanding flat FLRW universe.

3.3 Conclusions

In this essay we have demonstrated the similarty between general relativity and some analogue that can be used to describe its behaviour at least in part. We began with simple examples such as geometrical acoustic, physical acoustic or vortex geometry, and lead up to advance versions based on BECs. In this essay we used abstract mathematics to bring us to the main idea of an analogies model. The reason for developing these analogue was to help us understand more about general relativity, and also to find a way to incorporate quantum gravity through understanding the basic idea of a quantum model, or that more about the condensate matter system. A possible extension of this work would be to pave the way for the development of technology to test the creation of cosmological particles in the lab.

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شكر وَ عَرفَانّ

اَلحَمد لِلَّه الذِي وَفَقنَا بِنعَمَتِه عَلي انّ نَتمِ هَذَا العَمَل المتوَاضعَ . وَ مِنّ هُنَا اودُ أَن اشكُر الّذينَ سَاعَدُنيِ عَلي اكمَال هَذَا البحَث التكميلي لنيل شِهَادة الدِبلوم العَالِي.

شكري موصول الي دكتور سَام ، وَ مُشرفي دكتور جيف مورغَان ، اَنَاهيتَا ، وَ خَالِص شُكرِي أَلي اسرة المَعَهَد

Bibliography

[BLV06] Calos Barcelo, Stefano Liberati, and Matt Visser, *Analogue gravity*, Granada, Spain, 2006, Unpublished manuscript.

[Cha00] C. J. Chapman, *High speed flow*, Cambridge Unversity, USA, 2000.

[LM.68] Milne-Thomson LM., *Theoretical hydrodynaic*, MacMillan, London, 1968.

[SA73] Misner C W; Thorne K S and Wheeler J A, *Gravitation*, San Francisco, 1973, Freeman.