Algebras over Monads

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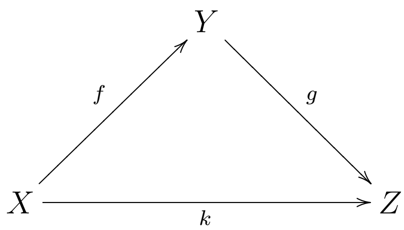
Abstract

The purpose of the project is to study some basic categorical concepts related to adjoint functors and monads, to explain how the classical algebraic structures can be regarded as Eilenberg-Moore algebras over monads, and to show how this general approach simplifies and allows to generalise various constructions of universal algebra.

Contents

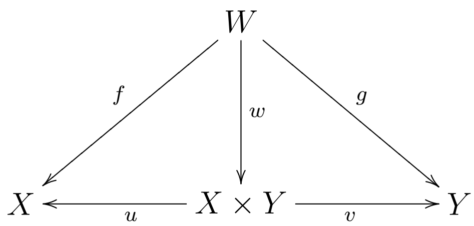
1. Introduction

Categorical theory has its origin from algebraic topology. It has much to do with the presentation of properties using arrows and diagrams. One usually says that what is important about any class of mathematical structures is the structure preserving maps between objects in the class. Thus, in category theory, morphisms are considered to be more important than the object as they also carry the infomation about the objects. It is convenient to represent a map from the set X to the set Y by an arrow $f : X \to Y$. Such an arrow assigns to an element $x \in X$ an element $f(x) \in Y$. Suppose we are considering sets as our structures, let $f : X \to Y$ and $g : Y \to Z$ be maps between sets. Then there is a composite $gf = g \circ f : X \to Z$. If for $f : X \to Y$ we have another function $h : Y \to X$ with $fh = 1_Y$ and $hf = 1_X$, where $1_X, 1_Y$ are identity maps of X and Y respectively, then we say that f is injective and surjective or that f is bijective. These sets and functions can be represented in a diagram as follows:



This diagram is said to be commutative if the arrow k is the composition of g and f, thus $k = gf : X \to Z$. These diagrams have a lot of meaning in category theory; hence, our essay will contain many of them. In the category of groups, X, Y, Z represents groups and f, g, k represents group homomorphisms.

Many properties of mathematical constructions may be represented by universal properties of diagrams [Lan98]. As an example let us consider the Cartesian product, $X \times Y$, of two sets X and Y. This product has projections $u : X \times Y \to X$ and $v : X \times Y \to Y$. If we consider a function $w : W \to X \times Y$ from another set W then it is uniquely determined by the composites uw and vw. Suppose we have $f : W \to X$ and $g : W \to Y$, then there exist a unique function $w : W \to X \times Y$ such that the following diagram commutes:



In this case the pair $\langle u, v\rangle$ is *universal* among the pairs from a set W to the sets X and Y and any pair $\langle f, g\rangle$ factors uniquely through the pair $\langle u, v\rangle$. This property characterises the cartesian product up to isomorphism; and in fact the same property and the characterisation holds when we replace sets, maps and Cartesian product by groups, homomorphisms and direct products respectively.

Universal properties can also be expressed as adjoints. If we denote the set of all functions between two sets W and X by $Hom(W,X)$; and the pairs of functions $f: U \to X$ and $g: V \to Y$ by $Hom(\langle U,V\rangle,\langle X,Y\rangle)$, then the correspondence $w \mapsto \langle uw, vw\rangle = \langle f,g\rangle$ indicated in the diagram above is a bijection:

$$Hom(\langle W, X \times Y\rangle) \cong Hom(\langle W,W\rangle,\langle X,Y\rangle)$$

and in fact it is natural. This bijection involves two constructions, $W \mapsto W$ sending each set to a diagonal pair $\langle W,W\rangle$ and $\langle X,Y\rangle \mapsto X \times Y$ sending a pair of sets to its cartesian product. In this case the the construction $X \times Y$ is referred to as the right adjoint to the construction $\langle W,W\rangle$ and $\langle W,W\rangle$ the left adjoint to the product.
The construction "Cartesian product" is a "functor", to be discussed in the next Chapter, since it applies suitably to the sets and the maps between them.

This project will take you through some concepts of category theory. In the first chapter we introduce concepts such as categories, functors, natural transformations and adjunctions. Chapter two talks briefly about Universal algebra and will introduce the notion of Ω-algebras, theirs examples and the free Ω-algebras. The final chapter takes you through the notion of a monad and its algebras in a category and will try to link these algebras to the universal algebras.

2. Categorical concepts

2.1 Sets and Classes

One of the main objectives of category theory is to discuss properties of totalities of mathematical objects such as the "set" of all groups or the "set" of all homomorphisms between any two groups.[Lan98] Such big collections are called **classes**, thus a class has elements that are classes in their own right. If a class A is an element of another class B then we say that A is a **set**. A class A is called a **subclass** of a class B if every element in A is an element of B. The axiom of pair formation states that to any two sets X and Y, there exist a set $\{X, Y\}$ with exactly the members X and Y. This allows us to form ordered pairs, triples and n-tuples. We can also form a class of all those sets which have a given property. For example, given two sets X and Y we can construct the classes:

$$X \cap Y = \{a | a \in X \text{ and } a \in Y\};$$

$$X \cup Y = \{a | a \in X \text{ or } a \in Y\};$$

$$X \times Y = \{(a, b) | a \in X \text{ and } b \in Y\}.$$

The last class $X \times Y$ is the usual product of two classes X and Y that we saw earlier.
In this Chapter, we will consider classes with a given property or structure.

2.2 Categories

Definition 2.1. A category $\mathbf{C}$ consists of the following:

- a class of objects in $\mathbf{C}$ denoted $\mathbf{ObC}$;
- for each $A, B \in \mathbf{ObC}$, a collection $Hom_{\mathbf{C}}(A, B)$ whose elements are maps or morphisms or simply arrows between A and B;
- for each $A \in \mathbf{ObC}$, a morphism $id_A \in Hom(A, A)$, called the identity;
- for each $A, B, C \in \mathbf{ObC}$, and for some morphisms $f \in Hom(A, B)$ and $g \in Hom(B, C)$, a function, $Hom(B, C) \times Hom(A, B) \to Hom(A, C)$, called the composition

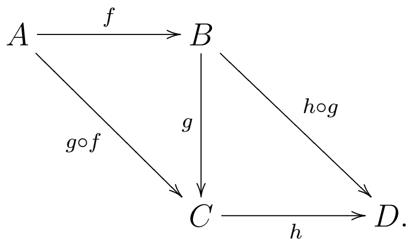
such that for all A, B, C, D in $\mathbf{ObC}$, $f \in Hom(A, B)$, $g \in Hom(B, C)$ and $h \in Hom(C, D)$,

- $id_B \circ f = f = f \circ id_A$;
- $(h \circ g) \circ f = h \circ (g \circ f)$.

The **dual** or **opposite category**, denoted by $\mathbf{C}^{op}$, is defined by reversing the morphisms of $\mathbf{C}$. This means that the objects of $\mathbf{C}^{op}$ are the objects of $\mathbf{C}$ and the morphisms from A to B in $\mathbf{C}$ are morphisms from B to A in $\mathbf{C}^{op}$. We give some examples of categories in the following subsection.

Examples of categories

1. **Set** : In the category of sets, the objects are the sets and the morphisms are the maps between the sets. To check this, let A and B be two sets and $f : A \to B$ a function between them. Then f is defined on all A and has all its images in B, thus $range(f) \subseteq B$. If we have another function $g : B \to C$, then there is a composition $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$ for $a \in A$. This composition is associative for if $h : C \to D$ is another function, we have the following diagram:



From this diagram we have, for any $a \in A$

$$((h \circ g) \circ f)(a) = h(g(f(a))) = (h \circ (g \circ f))(a),$$

hence, $(h \circ g) \circ f = h \circ (g \circ f)$. Finally every set A has an identity function $id_A : A \to A$ and we have, for $f : A \to B$, $f \circ id_A = id_B \circ f$. Hence sets form a category.

2. **Grp** : This is the category of groups which has objects as groups and the morphisms are the group homomorphisms. Composition of homomorphisms in groups is done in the same way as in sets.

3. **Rng** : In this category the objects are rings and the morphisms are the ring homomorphisms.

4. **Top** : This is a category of topological spaces in which the objects are the topological spaces and the morphisms are the continuous maps between them.

5. We can form a new category by taking the product of two categories. If $\mathbf{C}$ and $\mathbf{D}$ are categories, then their product is denoted as $\mathbf{C} \times \mathbf{D}$. This new category has objects as pairs (A, B) for A in $\mathbf{C}$ and B in $\mathbf{D}$. The morphisms are also pairs $(f, g) : (A, A') \to (B, B')$, where $f : A \to A'$ in $\mathbf{C}$ and $g : B \to B'$ in $\mathbf{D}$. Composition is defined componentwise, i.e. $(f', g') \circ (f, g) = (f' \circ f, g' \circ g)$ and the identity $(1_{\mathbf{C}}, 1_{\mathbf{D}}) = 1_{(\mathbf{C},\mathbf{D})}$

If all the morphisms of the category are identity morphisms we call such a category a **discrete category**. Categories with one object are referred to as **monoids**.
A monoid M is defined as a set equipped with a binary operation $\mu : M \times M \to M$ and a unit element η in M such that for all a, b, c in M we have:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \qquad \text{and} \qquad a \cdot \eta = a = \eta \cdot a\ .$$

2.3 Properties of Morphisms

Let A, B and C be objects of any category $\mathbf{C}$.
A morphism $f : B \to C$ for objects B and C is called a **monomorphism** in $\mathbf{C}$ if for any pair of arrows $g, h : A \to B$, the equality $f \circ g = f \circ h$ imply that $g = h$. We also say that f is **monic**.

A morphism $f : B \to C$ is called an **epimorphism** or **epic** in $\mathbf{C}$ if for any pair of arrows $g, h : C \to D$, the equality $g \circ f = h \circ f$ imply that $g = h$.
An epimorphism in the category $\mathbf{C}$ forms a monomorphism in the dual category $\mathbf{C}^{op}$, thus one can say that an epimorphism is a dual of the monomorphism.

A morphism $f : A \to B$ is called an **isomorphism** in $\mathbf{C}$ if there exists another morphism $g : B \to A$ such that $gf = id_A$ and $fg = id_B$. The morphism g is called the inverse of f and is denoted by f^{-1}. In fact when g exists, it is unique.
If we consider a group as a category, every morphism is an isomorphism [Wei94]. Isomorphism in the category of sets is a set bijection and in the category of topological spaces is a homeomorphism.

An **endomorphism** is an arrow $f : A \to A$ from an object onto itself. If $g : A \to A$ is another endomorphism, then the composite $g \circ f$ is also an endomorphism.

Definition 2.2. A **subcategory** $\mathbf{E}$ of a category $\mathbf{C}$ is any category whose objects and morphisms are contained is $\mathbf{C}$ and is also closed in $\mathbf{C}$ under the operations of domain, codomain, composition and identity of $\mathbf{C}$. We say that $\mathbf{E}$ is a **full** subcategory of $\mathbf{C}$ if for all objects $A, B \in \mathbf{E}$,

$$Hom_{\mathbf{E}}(A, B) = Hom_{\mathbf{C}}(A, B)$$

For example the category of abelian groups makes a full subcategory of groups.

2.4 Functors

Functors were first recognised in algebraic topology, where they arise naturally when geometric properties are described by means of algebraic invariants. And hence, they arise naturally in algebra. A functor is a morphism of categories. [LB67]

Definition 2.3 (Covariant functor)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories; a covariant functor $F : \mathbf{C} \to \mathbf{D}$ consists of the following:

- the object function F which assigns to each $A \in \mathbf{C}$ an object $FA \in \mathbf{D}$

- an arrow function which assigns to each arrow $f : A \to B$ in $\mathbf{C}$ a function $Ff : FA \to FB$ of $\mathbf{D}$

such that

- $F(1_A) = 1_{FA}$;
- $F(g \circ f) = Fg \circ Ff$; whenever $g \circ f$ is defined in $\mathbf{C}$.

As an example of a functor let us consider a power set. The power set of the set X is defined to be $PX = \{A|A \subseteq X\}$. Given a function $f : X \to Y$, we can define $Pf : PX \to PY$ to be the direct image operation: for all $A \in PX$, define

$$(Pf)(A) = f(A) = \{f(x)|x \in A\}$$

This assignment gives us a functor $P : Set \to Set$. To check that identities and composition are preserved we do the following:

$$\begin{aligned}(Pid_X)(A) &= \{id_X(x)|x \in A\}\\ &= \{x|x \in A\}\\ &= A\\ &= id_{PX}(A).\end{aligned}$$

$$\begin{aligned}(Pg \circ Pf)(A) &= (Pg)(\{f(x)|x \in A)\}\\ &= \{g(y)|y \in \{f(x)|x \in A\}\}\\ &= \{g(f(x))|x \in A\}\\ &= \{g \circ f(x)|x \in A\}\\ &= (P(g \circ f))(A).\end{aligned}$$

A **contravariant functor** is a functor L from a dual category of $\mathbf{C}$ to $\mathbf{D}$, i.e, a functor $L : \mathbf{C}^{op} \to \mathbf{D}$.

A functor is called a **forgetful functor** or **underlying functor** if it forgets some or all of the structure of the algebraic object. For example a functor $F : \mathbf{Grp} \to \mathbf{Set}$ is a forgetful functor for it forgets part of the structure of a group and some properties of the group homomorphisms. These forgetful functors will be useful in describing algebras over monads in Chapter 4.

From the definition of a functor, we have that any functor $F : \mathbf{C} \to \mathbf{D}$ induces a mapping

$$Hom_{\mathbf{C}}(A, B) \to Hom_{\mathbf{D}}(F(A), F(B)) \tag{2.1}$$

for $A, B \in \mathbf{C}$.
A functor F is said to be **faithful** if (2.1) is injective and is said to be **full** if (2.1) is surjective. Forgetful functors are usually faithful functors.
When a functor is both full and faithful we say that it is **fully faithful**. Note however that fully faithful does not imply isomorphism, for there may be some objects of $\mathbf{D}$ that are not images of F.

Definition 2.4. Let $\mathbf{C}$ and $\mathbf{D}$ be categories; a functor $F : \mathbf{C} \to \mathbf{D}$ is an isomorphism if there exist another functor $G : \mathbf{D} \to \mathbf{C}$ such that the composites $F \circ G = 1_{\mathbf{D}}$ and $G \circ F = 1_{\mathbf{C}}$, where $1_{\mathbf{C}}$ and $1_{\mathbf{D}}$ are identity functors of $\mathbf{C}$ and $\mathbf{D}$ respectively.

The functor which is an isomorphism is referred to as the **equivalence of the categories**.

Proposition 2.5. *Functors preserve isomorphism.*

Proof. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let A and A' be isomorphic objects in $\mathbf{C}$. Then there exist an isomorphism $f : A \to A'$ and this imply that there are maps $Ff : FA \to FA'$ and $Ff^{-1} : FA' \to FA$ in $\mathbf{D}$; where f^{-1} is the inverse of f. To show that Ff and Ff^{-1} are inverses of each other we do the following calculation:

$$Ff^{-1} \circ Ff = F(f^{-1} \circ f) = F(1_A) = 1_{FA}.$$

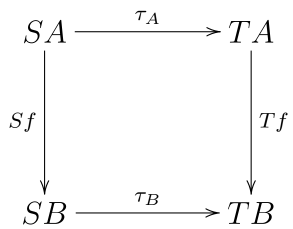
This completes the proof. ■

Functors can also be composed in the normal way like functions. Given two functors $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are categories; the composition of G and F is defined as $G \circ F : \mathbf{A} \to \mathbf{C}$. This composition is associative hence we can form a category which has objects small categories and morphisms functors between the categories with the identity functor $Id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$. This category is denoted as $\mathbf{Cat}$.

2.5 Natural Transformations

It is natural to think of mappings between functors. In this case we talk of the notion of natural transformations. We define the natural transformation as follows.

Definition 2.6. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $S, T : \mathbf{C} \to \mathbf{D}$ be functors.
A **natural transformation** of functors $\tau : S \to T$ is a function which assigns to each object A of $\mathbf{C}$ an arrow $\tau_A = \tau SA \to TA$ of $\mathbf{D}$ such that for every arrow $f : A \to B$ on $\mathbf{C}$ the following diagram commutes.



(2.2)

We say that $\tau_A : SA \to TA$ is natural in A whenever (2.2) holds, i.e, when $Tf \circ \tau_A = \tau_B \circ Sf$. It is not yet clear what this definition of natural transformation means. [Cam98]
The arrows τ_A,τ_B,$\tau_C, \ldots\ldots\ldots$ are called the components of the natural transformation τ. If the components of the natural transformation τ are invertible, i.e. have inverses, we call τ a natural equivalence or a natural isomorphism; in symbol $\tau : S \cong T$.

An example of a natural transformation is the singleton operation $s_X : X \to PX$, where $s_X(x) = \{x\}$ and PX is the power set of X. This operation is defined for any set X and it is natural in X, i.e. for all $f : X \to Y$, the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{s_X} & PX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Pf} \\ Y & \xrightarrow{s_Y} & PY. \end{array}$$

To prove this let $x \in X$, then

$$\begin{aligned} (Pf \circ s_X)(x) &= (Pf)(s_X(x)) \\ &= (Pf)(\{x\}) \\ &= \{fx\} \\ &= s_Y(fx) \\ &= s_Y \circ f(x). \end{aligned}$$

Therefore, $s_X : X \to PX$ is a natural transformation in the parameter X.

2.6 Adjunctions

The concept of adjoint functors was formulated by Daniel Kan. This concept provides a different formulation of properties of free objects and their universal constructions. The notion of adjoint functors is so important to Universal Algebra because it includes the concept of the free algebras which is in general extended to infinitary algebras [Coh81]. We begin with the concepts of initial and final or terminal objects in a category.

In any category $\mathbf{C}$, X is called the **initial** object of $\mathbf{C}$ if for any object A in $\mathbf{C}$ there is only one morphism from X to A. On the other hand Y is called the **final** or **terminal** object of $\mathbf{C}$ if for any object A in $\mathbf{C}$ there is exactly one morphism from A to Y. The terminal object in $\mathbf{C}$ is sometimes called the initial object of the opposite category $\mathbf{C}^{op}$. A category may have many initial objects but they are isomorphic. This is also true for terminal objects. Initial objects are used to describe universal functors.

Let us consider the forgetful functor U : **Grp** $\to$ **Set**. If we fix the set X, we can construct a category (X, U), called a **comma category**, whose objects are maps $X \to UG$ and the morphisms are commutative triangles arising from the homomorphism $f : G \to H$, where $G, H \in$

$\mathbf{Grp}$. A universal arrow from an object G of Grp to a functor $U : \mathbf{Grp} \to$ **Set** consists of an initial object in the comma category. The initial object in this category (X, U) is the free group FX with canonical map $X \to UFX$. The functor F is a universal functor, and is said to be the left adjoint of the forgetful functor U. This determines the natural equivalence or natural isomorphism:

$$Hom_{\mathbf{Grp}}(FX, G) \cong Hom_{\mathbf{Set}}(X, UG). \tag{2.3}$$

When (2.3) is true the pair (F, U) is called an $\mathbf{adjoint\ pair}$, and we say that F is the $\mathbf{left\ adjoint}$ of U and U the $\mathbf{right\ adjoint}$ of F. We now give a formal definition of an adjunction.

Definition 2.7. Let $\mathbf{C}$, $\mathbf{D}$ be categories and $F : \mathbf{C} \to \mathbf{D}$, $G : \mathbf{D} \to \mathbf{C}$ be functors, an $\mathbf{adjunction}$ from $\mathbf{C}$ to $\mathbf{D}$ is a triple $\langle F, G, \varphi \rangle : \mathbf{C} \to \mathbf{D}$, where φ is a function which assign to each pair of objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$ a bijection of sets

$$\varphi = \varphi_{X,Y} : Hom_{\mathbf{D}}(FX, Y) \cong Hom_{\mathbf{C}}(X, GY) \tag{2.4}$$

which is natural in the variables X and Y.

If we set $Y = FX$ in (2.4), we get the identity 1_{FX} on the left corresponding to the natural transformation $\eta_X : X \to GFX$ on the right. In this case η_X is the image of 1_{FX} under φ, thus $\eta_X = \varphi(1_{FX})$, and is called the $\mathbf{unit}$ of the adjunction. We also have that for any morphism $f : X \to GY$ of a $\mathbf{D}$-object Y can be factored uniquely by η as $f = \eta \cdot Gf'$, where $f' : FX \to Y$ corresponds to f under (2.4).
In the same way, if we set $X = GY$ in (2.4), we get the identity 1_{GY} on the right corresponding to a morphism $\varepsilon_Y : FGY \to Y$ which is its image under φ^{-1}. The transformation ε is called the $\mathbf{counit}$ of the adjunction; and for any object Y of $\mathbf{D}$, $g : FX \to Y$ can also be factored uniquely by ε as $g = Fg' \cdot \varepsilon$, for $g' : X \to GY$ which correspond to g in (2.4).
This determines the adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{D}$ and also determines the isomorphism in (2.4). We observe that $f : X \to GY$ determines $Ff : FX \to FGY$ and hence $Ff \cdot \varepsilon : FX \to Y$; and $g : FX \to Y$ gives rise to $\eta \cdot Gg : X \to GY$.

In most cases the left adjoint of the forgetful functor turns out to be something like a free algebra, while the the right adjoint when it exist usually describes a subset with some closure property.[Coh81]

Examples of Adjunctions

1. Given any set X and a fixed field K, there is a vector space $F(X)$ whose basis is X and the elements are the linear combinations $\Sigma_i \lambda_i x_i$ of elements of $x_i \in X$ and $\lambda_i \in K$. It has the universal property that any function from X to the vector space V extends uniquely to the linear map

   $$F(X) \to V.$$

   If we consider a functor $U : \mathbf{Vct} \to \mathbf{Set}$, then for any set X and a vector space V there is a restriction:

   $$\mathbf{Vct}(F(X), V) \to \mathbf{Set}(X, U(V)).$$

This is a bijection, hence we have functors $\mathbf{Vct} \underset{F}{\overset{U}{\rightleftarrows}} \mathbf{Set}$ such that for each $X \in \mathbf{Set}$ and $V \in \mathbf{Vct}$

$$Hom_{\mathbf{Vct}}(F(X), V) \cong Hom_{\mathbf{Set}}(X, U(V)).$$

This gives us the required adjunction.

2. Consider the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$. Its left adjoint F : **Set** $\to \mathbf{Grp}$ maps an object X in $\mathbf{Set}$ to FX in $\mathbf{Grp}$. In the same way as in Example 1 we have the adjunction.

3. We also have an adjunction formed by a forgetful functor $U : \mathbf{Grp} \to \mathbf{Mon}$ and its left adjoint functor $F : \mathbf{Mon} \to \mathbf{Grp}$.

2.7 Transformations of Adjoints

Is there any map between two adjunctions? In this section we answer this question by a brief discussion on transformations of adjoints.

Definition 2.8. Let $\langle F, G, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{D}$ and $\langle F', G', \eta', \varepsilon' \rangle : \mathbf{C}' \to \mathbf{D}'$ be two adjunctions. A "*map of adjunctions*" from the first adjunction to the second adjunction is a pair of functors $K : \mathbf{D} \to \mathbf{D}'$ and $L : \mathbf{C} \to \mathbf{C}'$ such that both the G and the F squares of functors in the diagram below commute,

$$\begin{array}{ccc} \mathbf{D} & \underset{F}{\overset{G}{\rightleftarrows}} & \mathbf{C} \\ {\scriptstyle K}\big\downarrow & & \big\downarrow{\scriptstyle L} \\ \mathbf{D}' & \underset{F'}{\overset{G'}{\rightleftarrows}} & \mathbf{C}', \end{array} \tag{2.5}$$

i.e. $LG = G'K$ and $KF = F'L$; and such that the diagram of Hom-sets and adjunctions below commutes for $X \in \mathbf{C}$ and $Y \in \mathbf{D}$

$$\begin{array}{ccc} Hom_{\mathbf{D}}(FX, Y) & \xrightarrow{\varphi} & Hom_{\mathbf{C}}(X, GY) \\ {\scriptstyle K=K_{FX,Y}}\big\downarrow & & \big\downarrow{\scriptstyle L=L_{X,GY}} \\ Hom_{\mathbf{D}'}(KFX, KY) & & Hom_{\mathbf{C}'}(LX, LGY) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ Hom_{\mathbf{D}'}(F'LX, KY) & \xrightarrow{\varphi'} & Hom_{\mathbf{C}'}(LX, G'KY) \end{array} \tag{2.6}$$

where $K_{FX,Y}$ is the map $f \mapsto Kf$ given by the functor K applied to $f : FX \to Y$.

Proposition 2.9. *Given two adjunction* $\langle F, G, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{D}$ *and* $\langle F', G', \eta', \varepsilon' \rangle : \mathbf{C}' \to \mathbf{D}'$, *and two functors* K *and* L *satisfying* (2.5)*, then the condition* (2.6) *of Hom-sets is equivalent to* $L\eta = \eta' L$*; and also* $\varepsilon' K = K\varepsilon$*.*

Proof. Suppose that the diagram (2.5) is commutative and let us set $Y = FX$ in (2.6), then we have the units $\eta : 1 \to GF$ and $\eta' : 1 \to G'F'$. This gives us

$$L\eta : L \to LGF \qquad \text{and} \qquad \eta' L : L \to G'F'L \,.$$

But from (2.5), we have $LGF = G'F'L$, hence, $L\eta = \eta' L$.
Dually, if we let $X = GY$ in (2.6), we get the counits ε and ε' and the equality

$$K\varepsilon : KFG \to K \qquad \text{and} \qquad \varepsilon' K : F'G'K \to K \,\cdot$$

Hence, $K\varepsilon = \varepsilon' K$.
Conversely, given that $L\eta = \eta' L$ and $K\varepsilon = \varepsilon' K$, then by the definition of φ and φ' by their units and the counits, we get (2.6). ■

3. Algebras

In this chapter we will briefly discuss Universal Algebras. We will describe the concept of Ω-algebras and their examples, subalgebras and free algebras. One of the aims of universal algebra is to extract, whenever possible, the common elements of several seemingly different types of algebraic structures.[BS81]

3.1 Ω-Algebras

Definition 3.1. Let A be a non-empty set and let $n \geq 0$ be an integer, then we write $A^0 = \emptyset$ and $A^n = \{x_1, x_2, ..., x_n | x_n \in A\}$. We define an n-**ary** as a function $\omega : A^n \to A$ where n is called the **arity** or **rank** of ω.
The function ω is *nullary* (constant) if its arity is 0; *unary* if arity is 1; *binary* if arity is 2 and *ternary* if arity is 3.

Definition 3.2. A **signature** of an algebra is a set Ω of function symbols such that $n > 0$ is assigned to each $\omega \in \Omega$, with n the arity of ω and ω the n-ary function symbol.
The subset of n-ary function symbol in Ω is denoted by Ω_n.

Definition 3.3. Let Ω be a signature, then an algebra A of type Ω is an ordered pair $\langle A, F\rangle$, where A is a non empty set and F is a family of finitary operations on A indexed by Ω, such that to each n-ary function $\omega \in \Omega$ there exist an n-ary operation ω_A on A. The set A in this case is called a **universe** or **underlying** set of $A = \langle A, F\rangle$, and the ω_A's are fundamental operations of A. Such an A is also called an Ω-**algebra**, sometimes written as A_Ω. The underlying set A is also called the carrier of A_Ω.

The underlying set A is unary if all function operations are unary; Mono-unary if it has one unary operation and is a groupoid or magma if it has one binary operation. The binary operation is usually denoted by $+$ or $\cdot$, and we write $a + b$ or $a \cdot b$ for the image of the pair $\langle a, b\rangle$ under this operation.

Examples of Ω-algebras

1. **Groups** : A group G is an Ω-algebra which satisfies the following axioms:

   $G1$. $(x * y) * z \approx x * (y * z)$;

   $G2$. $x * 1 \approx x \approx 1 * x$;

   $G3$. $x * x^{-1} \approx 1 \approx x^{-1} * x$;

   for all x, y, z in G.
   A group G is abelian or commutative if it satisfies $G1, G2, G3,$ and the following axiom:

   $G4$. $x * y \approx y * x$.

In this case Ω has 3 elements; a nullary operation, a binary operation and a unary operation.

2. **Magma** : This is a set G in which Ω has a single binary operation.
   - **Semigroups** : This is a magma in which $G1$ of the group axioms holds.
3. **Monoids** : A monoid is a set in which Ω has two elements a binary and a nullary operation satisfying the axioms $G1$ and $G2$ of the group.
4. **Quasigroups** : A quasigroup Q is an Ω-algebra in which Ω has three binary operations as its elements satisfying the following:

   $Q1.\ x\backslash(x \cdot y) \approx (y); \qquad (x \cdot y)/y \approx x\ ;$

   $Q2.\ x \cdot (x\backslash y) \approx y; \qquad (x/y) \cdot y \approx x\ .$
5. **Loops** : A loop is a quasigroup with identity which satisfies $Q1, Q2$ and $G2$.
6. **Rings** : A ring is an Ω-algebra R in which Ω has two binary operations ($+$ and $\cdot$), a nullary operation and a unary operation satisfying the following:

   $R1$. R is an abelian group with respect to $+$;

   $R2$. R is a semigroup with respect to $\cdot$;

   $R3.\ x \cdot (y + z) \approx x \cdot y + x \cdot z; \qquad (x + y) \cdot z \approx (x \cdot z) + (y \cdot z)\ .$
7. **Rings with the identity** : These are rings with a further property G2 of groups.
8. **Modules over rings** :
   - A left R-module M is an Ω-algebra with a binary, a unary and a nullary operation in which M is an additive abelian group and has a further unary operator $f_r : R \times M \to M$ which satisfies the following:
     For all $r, s \in R$ and $x, y \in M$ we have

     $M1.\ f_r(x + y) \approx f_r(x) + f_r(y);$

     $M2.\ f_{r+s}(x) \approx f_r(x) + f_s(x);$

     $M3.\ f_r(f_s(x)) \approx f_{rs}(x).$

     We call f_r the scalar action where f is a map which takes each $r \in R$ to its corresponding map f_r, and we write, for $x \in M$, $f_r(x) = rx$. Then $M1$ states that every f_r is a group homomorphism of M, and if R is a ring with identity, the other axioms assert that f is a ring homomorphism from R to the endomorphism ring $End(M)$. Thus a module is a ring action on an abelian group.
   - A right R-module satisfies $M1$ and $M2$ above, but $M3$ becomes

     $M3'.\ f_s(f_r(x)) \approx f_{rs}(x).$

3.2 Subalgebras and Homomorphisms

Definition 3.4. Let A and B be Ω-algebras, B is the **subalgebra** of A if the carrier of B is a subset of the carrier of A; and if $\omega \in \Omega$ define operations ω_A and ω_B in A and B respectively, then B admits ω_A and $\omega_A/B = \omega_B$ for each $\omega \in \Omega$.

Thus any subset of the Ω-algebra A can be defined in one way as the Ω-**subalgebra** of A.

Remark 3.5. The set of all subalgebras of an Ω-algebra A always contain A; and the subalgebra B is called proper if it is distinct from A.

Definition 3.6. Let A and B be Ω-algebras. A mapping $f : A \to B$ is a **homomorphism** from set A to set B if

$$f(\omega(a_1, ..., a_n)) = \omega(f(a_1), ..., f(a_n))$$

for every $n = 0, 1, 2, ..., \omega \in \Omega$ and $a_1, ..., a_n$ in A. The map f is called an **isomorphism** if it has an inverse $f^{-1} : B \to A$, and in this case, A is said to be isomorphic to B written as $A \cong B$.

From an algebraic point of view, isomorphic algebras can be regarded as equal or the same, as they would have the same algebraic structure, and would differ only in the nature of the elements; thus the phrase "*they are equal up to isomorphism*" is often used. [BS81]

Theorem 3.7. *Let A, B and C be Ω-algebras. Then*

1. *the identity map $1_A : A \to A$ is a homomorphism;*
2. *if $f : A \to B$ and $g : B \to C$ are homomorphisms, then their composite $gf = g \circ f$ is also a homomorphism.*

Proof. For every $n = 0, 1, 2, ..., \omega \in \Omega$, $a_1, ..., a_n$ in A and $b_1, ..., b_n$ in B we have:

$$\begin{aligned} 1_A(\omega(a_1, ..., a_n)) &= \omega(1_A(a_1), ..., 1_A(a_n)) \\ &= \omega(a_1, ..., a_n); \end{aligned}$$

and this proves (1).
To prove (2) we do the following calculation:

$$\begin{aligned} gf\omega(a_1, ..., a_n) &= g(\omega(f(a_1), ..., f(a_n)), \ \textit{since } f \textit{ is a homomorphism} \\ &= g(\omega(b_1, ..., b_n) \\ &= \omega(g(b_1), ..., g(b_n) \\ &= \omega(gf(a_1), ..., gf(a_n)). \end{aligned}$$

This completes the proof. ■

From this theorem we have that the Ω-algebras form a category in the same way as sets. The objects are the Ω-algebras, and the morphisms are the homomorphisms with the usual composition. The category of Ω-algebras is denoted by Ω-**Alg**. As we saw earlier, a category of groups (**Grp**), rings (**Rng**) and Monoids are examples of Ω-**Alg** (see Chapter 2).

3.3 Free algebras

Free algebras are determined by the cardinal of a free generating set.[Coh81]
Let Ω be a signature as above and let X be a set. We introduce a new signature $\Omega[X]$ as follows:

- $\Omega[X]_0$ is the disjoint union of Ω_0 and X;
- $\Omega[X]_n = \Omega_n$ for $n \geq 1$.

If C is a full subcategory of Ω-$\mathbf{Alg}$, we write $C[X]$ to denote the corresponding full subcategory in $\Omega[X]$-$\mathbf{Alg}$. We present this idea in a formal definition as follows:

Definition 3.8. Let A be a full subcategory of Ω-$\mathbf{Alg}$, then $A[X]$ is a full subcategory of $\Omega[X]$-$\mathbf{Alg}$ with objects all those $\Omega[X]$-algebras that are in A when we consider them as Ω-algebras.

We can also say that an $\Omega[X]$-algebra is a pair (A, f) in which A is an Ω-algebra and $f : X \to A$ an arbitrary map. In this case the $\Omega[X]$-algebra homomorphism $\alpha : (A, f) \to (A', f')$ is an Ω-algebra homomorphism $\alpha : A \to A'$ with $\alpha f = f'$.

Definition 3.9. Let A be a full subcategory of Ω-$\mathbf{Alg}$ and X a set. A **free algebra**, denoted by $F_A(X)$, in A over X is an initial object $F_A(X) = (F_A(X), \varphi)$ in the category of $A[X]$.

We will write $F(X)$ instead of $F_A(X)$. Note that φ maps X to $F(X)$ and is always injective. Since $F(X)$ is defined up to isomorphism, we can assume that $F(X)$ contains X as its subset so that $\varphi : X \to F(X)$ is an inclusion map. In this case we will say that X is the **canonical basis** of $F(X)$.

Theorem 3.10. *Suppose C contains all its subalgebras, i.e. is closed, and $\varphi : X \to F(X)$ is an injection. Then $F(X)$ is generated by the image of φ.*

Proof. Let $A = \langle \varphi(X) \rangle$ be a subalgebra of $F(X)$ generated by the image of φ. Then A is in C and is a $\Omega[X]$-algebra in $F(X)$. Thus we have the following morphisms in $C[X]$:

- the inclusion homomorphism $\alpha : A \to F(X)$;
- the homomorphism $\beta : F(X) \to A$, determined by the fact that $F(X)$ in an initial object.

We have also by the initiality of $F(X)$ that $\alpha\beta = 1_{F(X)}$; hence $A = F(X)$. ■

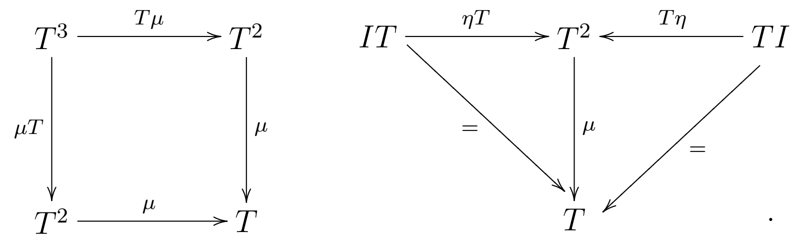
If A is in C and is an $\Omega[X]$-algebra in $F(X)$, we say that A is freely generated by X, i.e. it is canonically isomorphic to the quotient of $F(X)$, if the map $\alpha : F(X) \to A$ is injective. This means that, on one hand, we can freely choose any map from X to C and such a map extend uniquely to a map from A to C. On the other hand; the elements of A are combinations of elements of X in which the elements are as *independent* (free) as possible.

4. Monads and T-algebras

The objective of this chapter is to give the reader an understanding of monads and their associated algebraic structures. Monads first arose explicitly in Homological algebra in the late 1950s and early 1960s. The applications of monads to algebraic structures were initiated by the description of the algebras for a monad by Eilenberg and Moore in 1965 and the closely related work by Kleisli.
Monads were first known as standard constructions and later as triples. But in this Chapter we will stick to the term "monad".

4.1 Monads

Definition 4.1. A **monad** $T = \langle T, \eta, \mu \rangle$ in a category $\mathbf{C}$ consists of an endofunctor $T : \mathbf{C} \to \mathbf{C}$ and two natural transformations $\eta : 1_{\mathbf{C}} \to T$ and $\mu : T^2 \to T$ such that the following diagrams commute

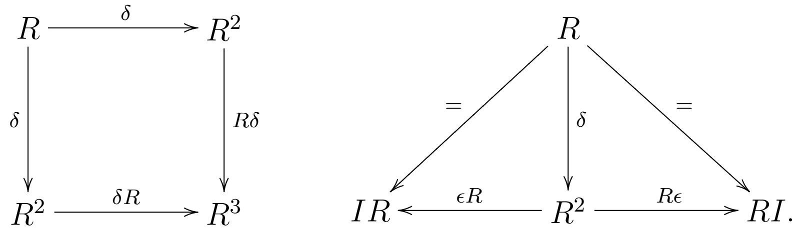


The first diagram represents the associative law and the second represent the left and the right unit laws for a monad.
This definition is the same as that of a monoid whose set of elements has been replaced by the endofunctor $T : \mathbf{C} \to \mathbf{C}$ and whose Cartesian product has been replaced by the composition of two functors, i.e., $\mu : M \times M \to M$ by $\mu : T^2 \to T$ and $\eta : 1 \to M$ by $\eta : 1_{\mathbf{C}} \to T$. The natural transformations η and μ are called the **unit** and the **multiplication** of the monad T respectively. In other words we can say that the monad in $\mathbf{C}$ is just a monoid in the category of endofunctors of $\mathbf{C}$ with the product $\times$ replaced by composition of endofunctors and the unit set by the identity endofunctor.[Lan98]
We can also define a comonad by reversing the arrows of the definition of a monad as follows:

Definition 4.2. A **comonad** in a category $\mathbf{D}$ consists of a endofunctor $R : \mathbf{D} \to \mathbf{D}$ and the natural transformations

$$\epsilon : R \to 1_{\mathbf{D}}; \qquad \delta : R \to R^2$$

such that the following diagrams are commutative



One can say that a comonad is a monad of the dual category $\mathbf{C}^{op}$.

As an example of a monad let us consider a preorder P. In a preorder an endofunctor $T : P \to P$ is a function $T : P \to P$ which is monotonic, that is, if $x \leqslant y$ in P then $Tx \leq Ty$. We have natural transformations η and μ when, for all $x \in P$,

$$x \leq Tx, \qquad T(Tx) \leq Tx \,. \tag{4.1}$$

The first equation of (4.1) gives $Tx \leq T(Tx)$ so that, if P is a partial order, $T(Tx) = Tx$. This gives a monad T as a closure operator in P which is a monotonic function $T : P \to P$, with $x \leq Tx$ and $T(Tx) = Tx$.

Remark 4.3. Every adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{A}$ gives rise to a monad in the category $\mathbf{C}$. The functors $F : \mathbf{C} \to \mathbf{A}$ and $G : \mathbf{A} \to \mathbf{C}$ have the composition $GF = T$, an endofunctor. The unit of the adjunction is the natural transformation $\eta : 1_{\mathbf{C}} \to T$, and the counit $\varepsilon : FG \to 1_{\mathbf{A}}$ gives rise to a natural transformation $\mu = G\varepsilon F : GFGF \to GF = T$. The associativity of μ is the commutativity of the diagram below

$$\begin{array}{ccc} GFGFGF & \xrightarrow{GFG\varepsilon F} & GFGF \\ {\scriptstyle G\varepsilon FGF}\big\downarrow & & \big\downarrow{\scriptstyle G\varepsilon F} \\ GFGF & \xrightarrow{G\varepsilon F} & GF. \end{array} \tag{4.2}$$

If we remove F from the front and G from behind in the diagram (4.2), we get the commutative diagram

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\varepsilon} & FG \\ {\scriptstyle \varepsilon FG}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ FG & \xrightarrow{\varepsilon} & I_A. \end{array}$$

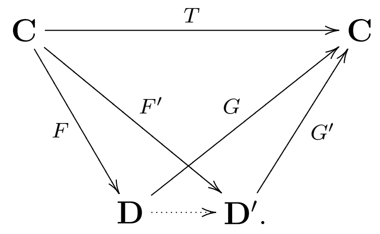
This corroborates with the definition of composition: $\varepsilon\varepsilon = \varepsilon.(FG\varepsilon) = \varepsilon.(\varepsilon FG)$. The unit laws are as in the diagram below:

$$\begin{array}{ccccc} I_C GF & \xrightarrow{\eta GF} & GFGF & \xleftarrow{GF\eta} & GFI_A \\ & {\scriptstyle =}\searrow & \big\downarrow{\scriptstyle G\varepsilon F} & \swarrow{\scriptstyle =} & \\ & & GF & & \end{array} .$$

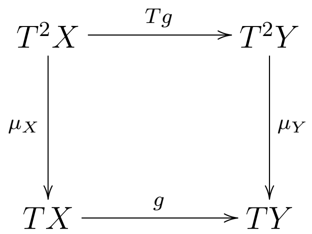
Thus $\langle GF, \eta, G\varepsilon F, \rangle$ is a monad in $\mathbf{C}$, and we will call this a monad defined by the adjunction $\langle F, G, \eta, \varepsilon \rangle$.

In the same way, each adjunction $\langle F, G, \eta, \varepsilon \rangle : \mathbf{B} \to \mathbf{D}$ defines a comonad $\langle FG, \varepsilon, F\eta G \rangle$ in $\mathbf{D}$.

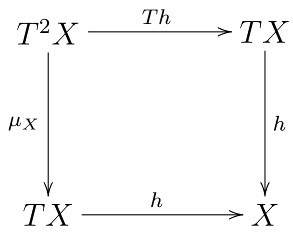
In general a monad $T = \langle T, \eta, \varepsilon \rangle$ can arise from different adjunctions and these adjunctions can be formed into a category [Coh81]. The objects of this category are then the adjunctions $\langle F, G, \eta, \varepsilon \rangle$ with $T = GF$, $\mu = G\varepsilon F$, represented as commutative triangles with the same base $T : \mathbf{C} \to \mathbf{C}$. The morphisms are the commutative tetrahedra consisting of two such triangles with the third vertex $\mathbf{D}$, $\mathbf{D}'$ respectively and a functor from $\mathbf{D}$ to $\mathbf{D}'$ which makes the diagram below commutative.

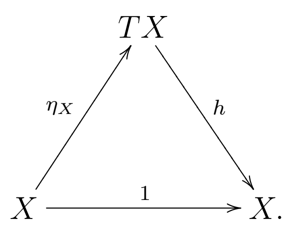


This adjunction category has an initial object $\langle F_T, G_T, \eta_T, \varepsilon_T \rangle : \mathbf{C} \to \mathbf{C}_T$ and a final object $\langle F^T, G^T, \eta^T, \varepsilon^T \rangle : \mathbf{C} \to \mathbf{C}^T$. In this case $\mathbf{C}_T$ has the same objects as $\mathbf{C}$, and the morphisms from X to Y in $\mathbf{C}_T$ are the morphisms $g : TX \to TY$ in $\mathbf{C}$ for which the following diagram commutes with the compositions as in $\mathbf{C}$.

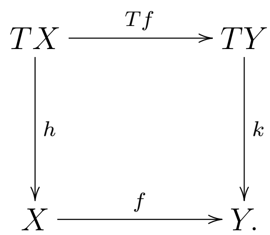


Composition in $\mathbf{C}_T$ is composition in $\mathbf{C}$ defined as follows: for $f : X \to Y$ and $g : Y \to Z$, $g \star f = \mu_Z \circ T(g) \circ f$. The functors F_T, G_T are defined by $F_T(X) = X$, $F_T g = Tg$, $G_T X = TX$, $G_T g = g$.
In the final object, the category $\mathbf{C}^T$ has as objects the pairs $\langle X, h \rangle$, where X is an object in $\mathbf{C}$ and $h : TX \to X$ is a structure map, such that the following diagrams commute;





The arrows $f : X \to Y$ in $\mathbf{C}$ are the morphisms of $\mathbf{C}^T$ from the pair $\langle X, h \rangle$ to $\langle Y, k \rangle$ which enable the diagram below to commute.



The initial caregory $\mathbf{C}_T$ is called the **Kleisli construction**, and the terminal category $\mathbf{C}^T$ is the **Eilenberg-Moore construction**; the objects of $\mathbf{C}^T$ are called the T**-algebras**.
This essay is meant to explore these T-algebras in relation to the universal algebras. We give a formal definition of the T-algebras and their morphisms in the section that follows.

4.2 Algebras for monads

We begin this section with some mathematical constructions which will lead us to T-algebras. We will use the monoidal approach to generate the T-algebras, and this will enable us to generalise to other algebras. This generalisation will then lead us to the conclusion that the T-algebras are the Universal algebras.

Definition 4.4 (Monoid actions)**.** Let M be a monoid with identity 1 and S be any set. An *action* of M on S is a function $h : M \times S \to S$ which satisfies the following axioms:

1. $1.s = s$
2. $(mm')s = m(m's)$

for all $m, m' \in M$ and $s \in S$.

In Definition 4.4, we have used the fact that $h(m, s) = ms$. In this case S is called an M-set. From this definition we have that the following diagrams are commutative:

$$\begin{array}{ccc} M \times M \times S & \xrightarrow{\mu \times 1_S} & M \times S \\ {\scriptstyle 1_M \times h}\downarrow & & \downarrow{\scriptstyle h} \\ M \times S & \xrightarrow{h} & S \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\eta \times 1_S} & M \times S \\ & {\scriptstyle =}\searrow & \downarrow{\scriptstyle h} \\ & & S \end{array} \tag{4.3}$$

where $\mu : M \times M \to M$, $\eta : 1 \to M$, 1_M and 1_S are identity maps of M and S respectively. To prove the commutativity of the first diagram in (4.3) we show that

$$h \circ \mu \times 1_S = h \circ 1_M \times h. \tag{4.4}$$

For all $m, m' \in M$ and $s \in S$, the left hand side of (4.4) has $(m, m', s) \mapsto (mm', s) \mapsto (mm')s$, while the right hand side has $(m, m', s) \mapsto (m, m's) \mapsto m(m's)$. But by axiom 2 of Definition 4.4, we have $(mm')s = m(m's)$. Also, the second diagram has $s \mapsto (1, s) \mapsto 1.s = s$; hence, diagrams (4.3) are commutative.

Now let us consider a mapping $\mathbf{Sets} \xrightarrow{M \times (-)} \mathbf{Sets}$, and let $f : X \to Y$ be any arbitrary map from the set X to the set Y. Let $\eta_X : X \to M \times X$, then $\eta : 1_{\mathbf{Sets}} \to M \times (-)$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & M \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1_M \times f} \\ Y & \xrightarrow{\eta_Y} & M \times Y \end{array}$$

is commutative. This is so because $x \mapsto (1, x)$ and $f(x) \mapsto (1, f(x))$ for $x \in X$, $f(x) \in Y$ and $1 \in M$.
In the same way, if we let $\mu_X : M \times M \times X \to M \times X$, then $\mu : M \times M \times (-) \to M \times (-)$ and the diagram below is commutative

$$\begin{array}{ccc} M \times M \times X & \xrightarrow{\mu_X} & M \times X \\ {\scriptstyle 1_M \times (1_M \times f)}\downarrow & & \downarrow{\scriptstyle 1_M \times f} \\ M \times (M \times Y) & \xrightarrow{\mu_Y} & M \times Y. \end{array}$$

Now, let $T(X) = M \times X$, then $T = M \times (-)$, $\eta : 1_{\mathbf{Sets}} \to T$ and $\mu : T^2 \to T$. Described in this way, T, μ and η define a monad $T = \langle T, \mu, \eta \rangle$, as we saw earlier.
Furthermore, diagrams (4.3) become:

$$\begin{array}{ccc} T^2(S) & \xrightarrow{\mu_S} & T(S) \\ {\scriptstyle T(h)}\downarrow & & \downarrow{\scriptstyle h} \\ T(S) & \xrightarrow{h} & S \end{array} \qquad \begin{array}{ccc} S & \xleftarrow{\eta_S} & T(S) \\ & \searrow^{=} & \downarrow{\scriptstyle h} \\ & & S \end{array} \qquad (4.5)$$

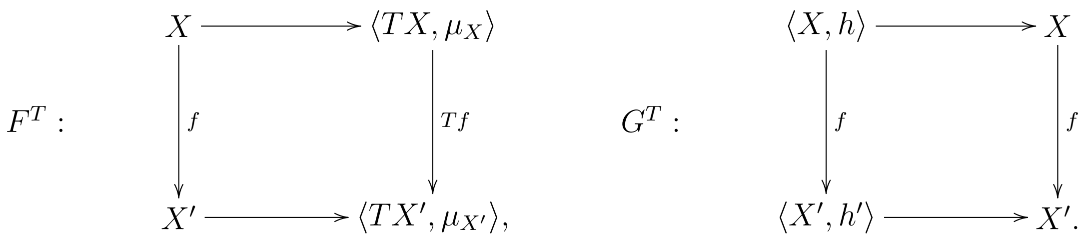
and this defines a T-algebra in the monad T. We now give a formal definition for T-algebras.

Definition 4.5. Let $T = \langle T, \eta, \mu \rangle$ be a monad in $\mathbf{C}$. A T-**algebra** is a pair $\langle S, h \rangle$ for an object $S \in \mathbf{C}$ and a map $h : TS \to S$ such that the diagrams (4.5) are commutative.
The first diagram represents the associative law and the second the unit law.
A morphism $f : \langle S, h \rangle \to \langle S', h' \rangle$ of T-algebras is an arrow $f : S \to S'$ of $\mathbf{C}$ such that the following diagram commutes

$$\begin{array}{ccc} TS & \xrightarrow{h} & S \\ {\scriptstyle Tf}\downarrow & & \downarrow{\scriptstyle f} \\ TS' & \xrightarrow{h'} & S'. \end{array}$$

Remark 4.6. Given any monad $T = \langle T, \eta, \mu \rangle$ on $\mathbf{C}$ and any object X of $\mathbf{C}$, the pair $\langle TX, \mu_X \rangle$ is a T-algebra. This pair is called the *free algebra* on the object X.

Theorem 4.7 (Every monad is defined by its T-algebras)**.** *If $T = \langle T, \eta, \varepsilon \rangle$ is a monad in $\mathbf{C}$, then the set of all T-algebras and their morphisms form a category $\mathbf{C}^T$. There is an adjunction $\langle F^T, G^T, \eta^T, \varepsilon^T \rangle : \mathbf{C} \to \mathbf{C}^T$ in which the functors F^T and G^T are given by the diagrams below*

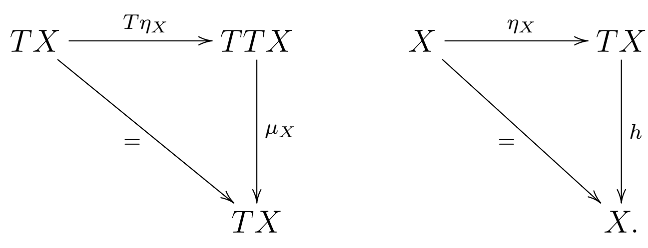


Also $\eta^T = \eta$ and $\varepsilon^T \langle X, h \rangle = h$ for each T-algebra $\langle X, h \rangle$.
The monad defined in $\mathbf{C}$ by this adjunction is the given monad $\langle T, \eta, \mu \rangle$

Proof. If $f : \langle X, h \rangle \to \langle X', h' \rangle$ and $g : \langle X', h' \rangle \to \langle X'', h'' \rangle$ are morphisms of the T-algebras, then their composition $gf : \langle X, h \rangle \to \langle X'', h'' \rangle$ is also a morphism. With this composition of the morphisms and the identity morphism $id_{\langle X,h \rangle} : \langle X, h \rangle \to \langle X, h \rangle$, the T-algebras form a category $\mathbf{C}^T$ as claimed. The functor G^T is the forgetful functor from $\mathbf{C}^T$ to $\mathbf{C}$ which forgets the structure map of each T-algebra. By Remark 4.6 for any object X in $\mathbf{C}$, the pair $\langle TX, \mu_X \rangle$, where $\mu_X : T(TX) \to TX$, is a T-algebra. Hence $X \mapsto \langle TX, \mu_X \rangle$ truly defines a functor $F^T : \mathbf{C} \to \mathbf{C}^T$. We also have that $G^T F^T X = G^T \langle TX, \mu_X \rangle = TX$, which implies that the unit η of the given monad is the natural transformation $\eta = \eta^T : 1_X \to G^T F^T$. Also $F^T G^T \langle X, h \rangle = F^T(X) = \langle TX, \mu_X \rangle$. From Definition 4.5 of a T-algebra, we have from the first diagram in (4.5) that the structure map $h : TX \to X$ is a morphism $\langle TX, \mu_X \rangle \to \langle X, h \rangle$ of T-algebras. From this we obtain the transformation

$$\varepsilon^T \langle X, h \rangle = h : F^T G^T \langle X, h \rangle \to \langle X, h \rangle,$$

which is natural by the definition of a morphism of algebras. The identities of the adjunction are as in the diagrams below:



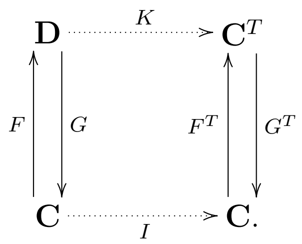
The first holds by the unit law for a monad T and the second by the unit law of the T-algebra. Therefore η^T and ε^T define an adjunction as claimed. It now remains to show that this adjunction determines the monad in $\mathbf{C}$.
It is clear that T is the endofunctor $G^T F^T$, the unit η^T is the original unit and the multiplication $\mu^T = G^T \varepsilon^T F^T$ is the original multiplication of T, for we have $\mu^T X = G^T \varepsilon^T \langle TX, \mu_X \rangle = G^T \mu_X = \mu_X$. ∎

4.3 Comparison with Algebras

Now we want to find out the relationship between the Eilenberg-Moore category and its original category. We do this by proving the theorem below. This will help us show that the familiar T-algebras are universal algebras. However, this demands that the comparison functor be an isomorphism. We will give some examples of algebras where the comparison functor is an isomorphism which will show us that the T-algebras are the universal algebras.

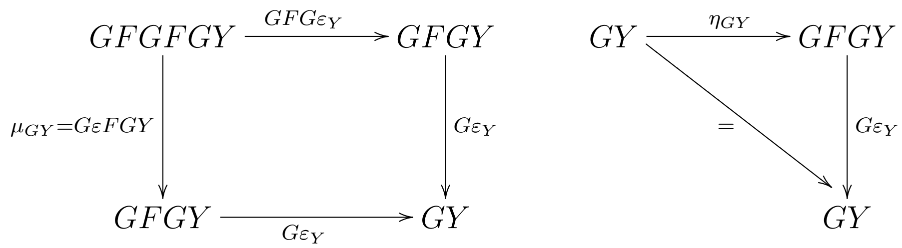
Theorem 4.8 (Comparison of Adjunctions with algebras)**.** *Let* $\langle G, F, \eta, \varepsilon \rangle : \mathbf{C} \to \mathbf{D}$ *be an adjunction and* $T = \langle GF, \eta, G\varepsilon F \rangle$ *the monad it defines in* $\mathbf{C}$*. Then there is a unique functor* $K : \mathbf{D} \to \mathbf{C}^T$ *with* $G^T K = G$ *and* $KF = F^T$*.*

Proof. The last part of the theorem suggests that we can draw the diagram below such that both the F square and the G square are commutative:



(4.6)

For each $Y \in \mathbf{D}$, the counit of the given adjunction defines an arrow $G\varepsilon_Y : GFGY \to GY$. We can consider this arrow as the structure map h of the T-algebra on the object $GY = X \in \mathbf{C}$ since the diagrams



are commutative. Then for any $f : Y \to Y'$, we can define K as

$$KY = \langle GY, G\varepsilon_Y \rangle, \qquad Kf = Gf : \langle GY, G\varepsilon_Y \rangle \to \langle GY', G\varepsilon_{Y'} \rangle. \tag{4.7}$$

The arrow Kf as defined is a morphism of T-algebras since ε is natural and commutes with $G\varepsilon$. Now, K is a functor by the fact that G is a functor. To show that $G^T K = G$ and $KF = F^T$ we do the following calculations: For all $X \in \mathbf{C}$, $Y \in \mathbf{D}$, $f \in \mathbf{C}$, $g \in \mathbf{D}$ and $h \in \mathbf{C}^T$, we have

$$G^T KY = G^T \langle GY, G\varepsilon_Y \rangle = GY; \text{ and } G^T Kg = G^T h = f = Gg;$$

thus, $G^T K = G$. Also,

$$KFX = KY = \langle GY, G\varepsilon_Y \rangle = \langle X, h \rangle = F^T X; \text{ and } KFf = Kg = F^T f;$$

implying that $KF = F^T$.
To show that K is unique, we first note that KY is a T-algebra and that $G^T K = K$ means that the underlying X-object of KY is GY. Hence $KY = \langle GY, h\rangle$ for some structure map h. Furthermore, $G^T K = K$ also means that the value of K acting on f in $\mathbf{D}$ is $Kf = Gf$, as in (4.7).
To determine h we first note that diagram (4.6) is commutative, and the two adjunctions $\langle F, G, ...\rangle$ and $\langle F^T, G^T, ...\rangle$ have the same unit. By Definition 2.8, we have that the functor $K : \mathbf{D} \to \mathbf{C}^T$ and the identity functor $I : \mathbf{C} \to \mathbf{C}$ define a map between the two adjunctions. Hence, by Proposition 2.9, we have that $K\varepsilon = \varepsilon^T K$. But K on arrows is G, so that $K\varepsilon_Y = G\varepsilon_Y$ for each $Y \in \mathbf{D}$. We also have, by the definition of ε^T, that $\varepsilon^T KY = \varepsilon^T\langle GY, h\rangle = h$. Thus $K\varepsilon = \varepsilon^T K$ imply that $G\varepsilon_Y = h$, so the structure map h is determined and hence K is unique. ■

We say that G is monadic if the comparison functor K is an isomorphism, i.e., if there exists another functor $L : \mathbf{C}^T \to \mathbf{D}$ such that $LK = I_{\mathbf{D}}$ and $KL = I_{\mathbf{C}^T}$.

We now give examples which show that the T-algebras for the familiar monads are the familiar universal algebras.

1. **Closure** : We already saw that a closure operation T on a preorder P is a monad. A T-algebra in a preorder P is $X \in P$ with $X \leq TX \leq X$, and the structure map $TX \leq X$. If P is a partial order then $X = TX$; hence, we have an element of the partial order $X \in P$, which is closed in the usual sense, as a T-algebra.

2. **Group actions** : The definitions

$$TX = G \times X, \qquad X \to G \times X, \qquad G \times (G \times X) \to G \times X,$$

$$x \mapsto \langle u, x\rangle, \qquad \langle g_1, \langle g_2, x\rangle\rangle \mapsto \langle g_1 g_2, x\rangle,$$

for $x \in X; g_1, g_2 \in G$ and u the unit element of G, define a monad $T = \langle T, \mu, \eta\rangle$ on $\mathbf{Set}$. A T-algebra is a set X with the structure map $h : G \times X \to X$ such that

$$h(u, x) = x, \qquad h(g_1 g_2, x) = h(g_1, h(g_2, x)).$$

If we write $g.x$ for $h(g, x)$ then we get the usual condition that $\langle g, x\rangle \mapsto g.x$. This defines an action of a group on a set X. The fact that the T-algebras are just the group actions follows from the mathematical construction using a monoid in Section 4.2.

3. **Modules** : Let R be a small ring and A a small abelian group. Then the following definitions

$$TA = R \otimes A, \qquad A \to R \otimes A, \qquad R \otimes (R \otimes A) \to R \otimes A,$$

$$a \mapsto 1 \otimes a, \qquad r_1 \otimes (r_2 \otimes a) \mapsto r_1 r_2 \otimes a,$$

for $r_1, r_2 \in R; a \in A$ define a monad on abelian groups. Following a similar process to the previous example, we have the T-agebras as the left R-modules.

In these examples the comparison functor is an isomorphism, and we can conclude that, "up to isomorphism", the T-algebras are Universal Algebras.

4.4 Free Algebras for a Monad

Let $\langle F, G, \varphi \rangle : \mathbf{C} \to \mathbf{D}$ be an adjunction and $A \in \mathbf{D}$ be any full subcategory which contains all the objects FX for $X \in \mathbf{C}$. Then we have another adjunction $\langle F_A, G_A, \varphi_A \rangle : \mathbf{C} \to A$ where the functor F_A is just F with its codomain restricted to A and G_A is G with its domain restricted to A. For $X \in \mathbf{C}$ and $Y \in A$ the given adjunction gives rise to a bijection

$$\varphi_A : Hom_A(F_A X, Y) = Hom_{\mathbf{D}}(FX, Y) \cong Hom_{\mathbf{C}}(X.GY) = Hom_{\mathbf{C}}(X, G_A Y)$$

which is natural in X and Y. In fact this new adjunction φ_A defines a monad in $\mathbf{C}$ as the first did. This tells us that a monad can arise from different adjunctions as was stated earlier in this Chapter. The smallest such adjunction is the one where $A = F\mathbf{C}$, the full subcategory of $\mathbf{D}$ with objects the free objects $FX \in \mathbf{D}$. We can construct the subcategory $F\mathbf{C}$ and the adjunction φ_A directly from the monad. This gives us a category instead of a subcategory, and it is the Kleisli category we described earlier in this Chapter. Hence, the Kleisli category of a monad T is equivalent to the full subcategory of free T-algebras of the monad.[BW99]

5. Conclusion

This essay has taken us through the basic concepts of categorical theory starting from the categories to the notion of adjunctions and their transformations. We have seen how morphisms play an important role in categorical theory. We saw how they preserve the object's structure.

We also looked at monads and their T-algebras. We saw how an adjunction gives rise to a monad, and that a monad can arises from different adjunctions. These adjunctions form a category whose initial and terminal objects are the Kleisli and the Eilenberg-Moore constructions, respectively. The Eilenberg-Moore construction contains the T-algebras which are the algebras for a monad and the Kleisli construction provides the free algebras for a monad. Between the Eilenberg-Moore category and its original category, there is a functor, called the comparison functor. However, this functor is not an isomorphism in all cases. For the T-algebras to be considered universal algebras, we need the comparison functor to be an isomorphism. We saw this isomorphism by obtaining the T-algebras from a monoid action on a set, where we obtained the T-algebras as M-sets. This fact led us to conclude that the T-algebras for familiar monads are just the universal algebras up to isomorphism.

Throughout this essay we have seen that we can have many mathematical constructions from the ones that exist. One would pursue this topic further by studying the Beck's theorem and also studying further the notion of equivalence of categories.

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