THEORETICAL ASPECTS OF CLASSICAL AND EXTENDED ELASTOPLASTIC MATERIAL MODELLING

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Abstract

The quasi-static evolution of an elastoplasticity body, with a single-surface constitutive law of kinematic and isotropic hardening type. This generalizes the classical small-strain elastoplasticity. This work presents the mathematical models, and proves existence and uniqueness of the solution, of the corresponding initial-boundary value problem. The analysis involves an explicit estimate for the effective ellipticity constant.

**Key words:** Elastoplasticity, kinematic, isotropic, hardening, rate independence, single-surface model, well-posedness, variational inequality.
Introduction

Developing a reliable model capable of predicting the behavior of structural systems represents one of the most challenging research tasks to face the analysts. The theory of elastoplasticity laid in the nineteenth-century, by TRESCA, ST.VENANT, LEVY, and BAUSCHINGER can be seen nowadays as a special case of the general theory of inelastic material behavior. The growing interest in this new area of research is due to the increasing complexity of the problems which have to be solved by the civil, naval, automobile and aeronautic engineers etc. Significant developments of the theory have been done in the early twentieth century by PRANDTL, REUSS and VON MISES (1913). However greater clarity in the mathematical framework by DRUCKER and PRAGER (1952); G. DUVAUT and J-L. LIONS (1976); and J.C. SIMO (1998) in the computational aspects. The recent works of HAN and REDDY [8] (1999), present the well-posedness of the classical elastoplastic model problem in the framework of convex and functional analysis, and the numerical aspects of the problem. In this work two equivalent formulations of the problem are described. The primal, whose point of departure is the flow law written in terms of the dissipation function, in which the unknowns are the displacement and the plastic strain fields, whereas the displacement and the stress fields are unknowns in the dual formulation.

This essay focuses first, on some results of the theoretical aspects of the small-strain theory of primal model of elastoplasticity, with hardening assumptions as presented in Han and Reddy [8]. Secondly, we will extend these results to new models of elastoplasticity. This work is divided into three chapters.

Chapter 1 contains the mathematical tools relevant to elastoplasticity theory, such as the functional formulation of the model and its associated analysis, and the interpretation of the normality law in the context of convex analysis.

Chapter 2 covers a variety of aspects of the theory. In the first part, we present the precise primal formulation of the initial-boundary value problem which governs the elasto-plastic behavior. In the second part, we consider the quasi-static case for small-strain elastoplasticity, with a single-surface law which is both linear kinematic and isotropic hardening type. In addition we prove the existence, uniqueness, and stability of its solution. Indeed, the existence of such solutions has been
obtained by Reddy and Han [8].

Chapter 3 is probably the most interesting one. It deals with new models for elastoplasticity. For a given yield function, we will find the dissipation function and further establish its well-posedness. These results are used to define the flow law and study the existence and uniqueness of the solution to the primal problem in the case of isotropic hardening.
The notations, the mathematical symbols and some common abbreviations used throughout this essay are listed below.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}$</td>
<td>set of integers,</td>
</tr>
<tr>
<td>$\mathbb{Z}^+$</td>
<td>set of positive integers,</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>set of real numbers,</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>$\mathbb{R} \cup {-\infty, +\infty}$,</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>open set in $\mathbb{R}^3$,</td>
</tr>
<tr>
<td>$\Gamma = \partial \Omega$</td>
<td>boundary of $\Omega$,</td>
</tr>
<tr>
<td>$\Gamma_D$</td>
<td>Dirichlet’s boundary,</td>
</tr>
<tr>
<td>$\Gamma_N$</td>
<td>Neumann’s boundary,</td>
</tr>
<tr>
<td>$n$</td>
<td>unit outer normal to $\Gamma$,</td>
</tr>
<tr>
<td>$I_A$</td>
<td>indicator function of $A$,</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>support function of $A$,</td>
</tr>
<tr>
<td>$f^*$</td>
<td>conjugate function to $f$,</td>
</tr>
<tr>
<td>$f^{**}$</td>
<td>biconjugate to $f$,</td>
</tr>
<tr>
<td>$\partial f(x)$</td>
<td>set the subgradients of at $x$,</td>
</tr>
<tr>
<td>$\frac{\partial f}{\partial x}$</td>
<td>partial derivative of $(.)$,</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$(\alpha_1, ..., \alpha_n)$,</td>
</tr>
<tr>
<td>$</td>
<td>\alpha</td>
</tr>
<tr>
<td>$D^\alpha$</td>
<td>$\frac{\partial^{(\alpha)}}{\partial x_1^{\alpha_1}...\partial x_n^{\alpha_n}}$,</td>
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<tr>
<td>$</td>
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<td>$</td>
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<tr>
<td>$</td>
<td>.</td>
</tr>
<tr>
<td>$\nabla$</td>
<td>gradient,</td>
</tr>
<tr>
<td>$div$</td>
<td>divergent,</td>
</tr>
<tr>
<td>$u$</td>
<td>displacement,</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>strain tensor,</td>
</tr>
</tbody>
</table>
\[ p \equiv \text{plastic strain tensor}, \]
\[ e \equiv \text{elastic strain tensor}, \]
\[ \xi \equiv \text{internal variable}, \]
\[ \chi \equiv \text{internal force}, \]
\[ \sigma \equiv \text{stress tensor}, \]
\[ \sigma_0 \equiv \text{yield stress}, \]
\[ \Sigma \equiv \text{generalized stress}, \]
\[ P \equiv \text{generalized plastic strain}, \]
\[ \dot{P} \equiv \text{generalized plastic rate}, \]
\[ \lambda \equiv \text{plastic multiplier}, \]
\[ \lambda, \mu \equiv \text{Lamé moduli}, \]
\[ M^3 \equiv 3 \times 3 \text{ symmetric matrices}, \]
\[ A^{-1} \equiv \text{inverse of } A, \]
\[ A^T \equiv \text{transpose of } A, \]
\[ S \equiv \text{spherical part of the stress}, \]
\[ \text{dev}(\sigma) = \sigma^D \equiv \text{deviatoric part of the stress } \sigma, \]
\[ \text{tr}(.) \equiv \text{trace of } (.), \]
\[ \text{meas}(.) \equiv \text{positive measure of } (.), \]
\[ C \equiv \text{tensor of elastic moduli}, \]
\[ f(\sigma, \chi) \equiv \text{yield function}, \]
\[ f(x, t) \equiv \text{body force}, \]
\[ S_n \equiv \text{surface force}, \]
\[ D_r, D \equiv \text{dissipation function}, \]
\[ : \equiv \text{scalar product of matrices}, \]
\[ . \equiv \text{scalar product of vectors}, \]
\[ i.e \equiv \text{that is}, \]
\[ a.e \equiv \text{almost everywhere}. \]
The goal of this chapter is to present some mathematical results that are relevant for the formulation and the understanding of the mathematical theory of elastoplasticity. These results are essentially from convex and functional analysis.

1.1 Results from convex analysis

In this section, our concern is to collect some definitions and results from convex analysis. More details and proofs of those results can be found in [1], [8], [6], and [13].

1.1.1 Convex sets, Convex functions

Let $X$ be a given Banach space (we consider only real linear space), and $X'$ its topological dual space.

**Definition 1.1** Let $Y$ be a subset of $X$,

(a) $Y$ is said to be convex if for any $x, y \in Y$ and for any $\theta \in (0, 1)$, $\theta x + (1 - \theta)y \in Y$.

(b) If $Y$ is a convex subset of $X$ and $x \in X$, the set:

$$N_Y(x) = \{x^* \in X' : <x^*, y - x> \leq 0 \quad \forall y \in X\}$$

(1.1)

is called the normal cone to the convex set $Y$ at $x$.

**Definition 1.2** Consider a function $f : X \to \mathbb{R}$, $(\mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\})$.

(a) $f$ is called a convex function if (and only if)

$$f((1 - \theta)x_1 + \theta x_2) \leq (1 - \theta)f(x_1) + \theta f(x_2) \quad \text{Jensen's inequality},$$

(1.2)
for all \( x_1, x_2 \in X \) and \( \theta \in (0, 1) \).

If Jensen’s inequality (1.2) is fulfilled without equality for all \( x_1 \neq x_2 \), then the function \( f \) is said to be strictly convex.

(b) The set defined by
\[
\text{dom}(f) = \{ x \in X : f(x) < \infty \}
\]
(1.3)
is called the effective domain of \( f \).

(c) The function \( f \) is said to be proper if \( \text{dom}(f) \neq \emptyset \) and \( f(x) > -\infty \) for all \( x \in X \).

(d) The function \( f \) is said to be positively homogeneous if:
\[
f(\alpha x) = \alpha f(x) \quad \text{for all} \quad x \in X \quad \text{and for all} \quad \alpha > 0.
\]
(1.4)

**Definition 1.3** Let \( A \subset X \).

(a) The indicator function \( I_A \) of \( A \) is defined by
\[
I_A(x) = \begin{cases} 
0 & \text{if } x \in A \\
+\infty & \text{if } x \notin A.
\end{cases}
\]
(1.5)

(b) The support function \( \sigma_A \) of \( A \) is defined on \( X' \) by
\[
\sigma_A(x^*) = \sup \{ <x^*, x> : x \in A \}.
\]
(1.6)

**Remark 1.1** \( A \) is a convex set if and only if \( I_A \) is convex (\( \text{dom}(I_A) = A \)).

1.1.2 Lower semi-continuity of (convex) function.

Let \( X \) be a Banach space.

**Definition 1.4** Consider a function \( f : X \to \overline{\mathbb{R}} \).

(a) A function \( f \) is said to be lower semi-continuous (l.s.c) at \( x_0 \) if, for \( \epsilon \in \mathbb{R} \) such that \( f(x_0) > \epsilon \), there exists a neighbourhood \( U(x_0) \) of \( x_0 \) such that \( f(x) > \epsilon \) for all \( x \in U(x_0) \).

(b) \( f \) is said to be sequentially lower semi-continuous at \( x_0 \) if and only if \( f(x_0) \leq \liminf_{n \to \infty} f(x_n) \)
holds for any sequence \( (x_n) \), \( x_n \in X \), satisfying \( \lim_{n \to \infty} x_n = x_0 \).

\[ \liminf_{n \to \infty} u_n = \sup_{n \in \mathbb{N}} \inf_{m \geq n} u_m. \]
Remark 1.2  
(i) If $f$ is continuous, then $f$ is lower semi-continuous.

(ii) For a Banach space, as assumed above, lower semi-continuity and sequentially lower semi-continuity are equivalent.

Definition 1.5  $f$ is said to be weakly lower semi-continuous at $x_0$ if for every sequence $(x_n)_{n \in \mathbb{N}}$ converging weakly to $x_0$, $f(x_0) \leq \liminf_{n \to \infty} f(x_n)$ holds.

Definition 1.6  A function $g : X \to [0, \infty]$ is called a gauge if

(i) $g(x) \geq 0 \quad \forall x \in X$,

(ii) $g(0) = 0$,

(iii) $g$ is convex, positively homogeneous, and l.s.c.

1.1.3 Conjugate functions, Subdifferential

Let $X$ be a reflexive Banach space, $X'$ its topological dual space, and the function $f : X \to \mathbb{R}$.

Definition 1.7  
(a) The function $f^* : X' \to \mathbb{R}$,

$$f^*(x^*) = \sup_{x \in X} \{ < x^*, x > - f(x) \}$$  \hspace{1cm} (1.7)

is called the conjugate function to $f$. (Sometimes also denoted as the Legendre-Fenchel conjugate)

(b) Let $f^* : X' \to \mathbb{R}$ be the conjugate function to $f : X \to \mathbb{R}$, then the function

$$f^{**}(x) = \sup_{x^* \in X'} \{ < x^*, x > - f^*(x^*) \}$$  \hspace{1cm} (1.8)

is called the biconjugate of the function $f$.

Proposition 1.1  If $f$ is proper, convex and l.s.c, then so is $f^*$, and

$$(f^*)^* \equiv f^{**} = f.$$  \hspace{1cm} (1.9)

Remark 1.3 If $A$ is a nonempty, convex and closed subset of $X$, its indicator function $I_A$ is proper, convex, l.s.c and
\[ I_{A}^{**} = \sigma_{A}^{*} = I_{A}. \] (1.10)
This follows from proposition 1.1.

Definition 1.8 Let $f$ be a proper function on $X$, then $x^* \in X'$ is said to be the subgradient of $f$ at $x \in \text{dom}(f)$ if
\[ f(y) - f(x) \geq <x^*, y - x> \quad \forall y \in X. \] (1.11)
The set of all the subgradients of $f$ at $x$ is called subdifferential and denoted by $\partial f(x)$, where
\[ \partial f(x) = \{x^* \in X' : f(y) - f(x) \geq (x^*, y - x) \quad \forall y \in X\}. \] (1.12)
The function $f$ is said subdifferendiable at $x$ if $\partial f(x) \neq \emptyset$.

Remark 1.4 (i) If $f \in C^1(\mathbb{R}^d)$ and convex then $\partial f(x) = \nabla f(x) \quad \forall x \in \mathbb{R}^d$.
(ii) $\partial I_A(x) = N_A(x), \text{ for all } x \in A$.

Theorem 1.1 Let $X$ be a reflexive Banach space, and let $f : X \longrightarrow \mathbb{R}$ be a proper, convex and weakly lower semi-continuous function. Given $x \in X$ and $x^* \in X'$, then $x^* \in \partial f(x)$ if and only if $x \in \partial f^*(x^*)$.

Proof See [13].

Remark 1.5 Theorem 1.1 is the useful result that permits to formulate the flow law in two equivalent forms, which are the primal and dual formulations of the mathematical model of elasto-plasticity.

1.2 Results from functional analysis and function spaces

1.2.1 Function spaces

Let $\Omega \subset \mathbb{R}^3$ be an open and bounded set.

The Sobolev spaces: $H^m(\Omega)$

Definition 1.9 Let $m$ be a nonnegative integer,
\[ H^m(\Omega) = \{v \in L^2(\Omega) : D^\alpha v \in L^2(\Omega), \text{ for any } \alpha \in \mathbb{Z}^d_+ \text{ with } |\alpha| \leq m\}, \] (1.13)
where the derivatives are taken in the distributions sense.
\( H^m(\Omega) \) is a Hilbert space endowed with the following inner product

\[
(u, v)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)}.
\]

(1.14)

In addition, \( H^m_0(\Omega) \) is the closure of \( C_0^\infty(\Omega) \) in \( H^m(\Omega) \), and is defined as follow

\[
H^m_0(\Omega) = \{ v \in H^m(\Omega) : v = 0 \text{ a.e on } \Gamma \}.
\]

(1.15)

**Theorem 1.2 (Korn’s first inequality)**

Let

\[
V = \{ v \in [H^1(\Omega)]^3 : v = 0 \text{ a.e on } \Gamma_0 \},
\]

(1.16)

where \( \Gamma_0 \) is a measurable subset of \( \Gamma \) with \( \text{meas}(\Gamma_0) > 0 \). There exists a constant \( c > 0 \) such that

\[
\| u \|_{[H^1(\Omega)]^3}^2 < c \int_\Omega |\varepsilon(u)|^2 \, dx \quad \forall u \in V \quad \text{where } \varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T).
\]

(1.17)

**Proof** See [8] Page 115.

**Spaces of vector-valued functions**

Let \( X \) be a Banach space and \( T \) a positive number.

**Definition 1.10** Let \( m = 0, 1, 2, \ldots \)

(a) **Spaces** \( C^m([0, T], X) \).

\[
C^m([0, T], X) = \{ v : [0, T] \rightarrow X \text{ such that, } v^{(k)} \text{ is continuous from } [0, T] \text{ to } X \text{ for } k \leq m \},
\]

(1.18)

where \( v^{(k)} \) is the \( k^{th} \) derivative with respect to \( t \).

It is a Banach space when endowed with the norm

\[
\| v \|_{C^m([0, T], X)} = \sum_{k=0}^{m} \max_{0 \leq t \leq T} \| v^{(k)}(t) \|_X.
\]

(1.19)

(b) **Spaces** \( L^p(0, T; X) \).

For \( 1 \leq p < \infty \), \( L^p(0, T; X) \) is the space of all measurable functions \( v \) from \( [0, T] \) to \( X \) for which

---

\( C_0^\infty(\Omega) \) is the set of function infinitely continuous with a compact support in \( \Omega \).
\[ ||v||_{L^p(0,T;X)} = \left( \int_0^T ||v(t)||_X^p \, dt \right)^{1/p} < \infty. \]  

For \( p = \infty \), the space, \( L^\infty(0,T;X) \) is the set of all measurable functions \( v \) from \([0,T]\) to \( X \) that are essentially bounded. Its norm is

\[ ||v||_{L^\infty(0,T;X)} = \text{ess sup}_{0 \leq t \leq T} ||v(t)||_X. \]

c) Let \( X' \) be the topological dual of a separable normed space \( X \). Then for \( 1 < p < \infty \) the topological dual space of \( L^p(0,T;X) \) is defined by

\[ [L^p(0,T;X)]' = L^q(0,T;X') \quad \frac{1}{p} + \frac{1}{q} = 1. \]

d) Space \( H^m(0,T;X) \) for \( m \geq 0 \) integer.

\[ H^m(0,T;X) = \{ f \in L^2(0,T;X) : f^{(i)} \in L^2(0,T;X) \quad i \leq m \}. \]

This is a Banach space with the norm

\[ ||f||_{H^m(0,T;X)} = \left( \sum_{i=0}^{m} ||f^{(i)}||_{L^2(0,T;X)}^2 \right)^{1/2}. \]

**Remark 1.6**

(i) The space \( L^p(0,T;X) \) with \( 1 \leq p \leq \infty \) is a Banach space when endowed with the norm \( ||v||_{L^p(0,T;X)} \).

(ii) If \( X \) is a Hilbert space with inner product \((.,.)_X\), then \( L^2(0,T;X) \) is a Hilbert space with the inner product

\[ (u,v)_{L^2(0,T;X)} = \int_0^T (u(t),v(t))_X \, dt. \]

(iii) If \( X \) is a Hilbert space, \( H^m(0,T;X) \) is also a Hilbert space with the following inner product

\[ (f,g)_{H^m(0,T;X)} = \int_0^T \sum_{i=0}^{m} (f^{(i)}(t),g^{(i)}(t))_X \, dt. \]

**Theorem 1.3** The embedding \( H^1(0,T;X) \hookrightarrow C([0,T];X) \) is continuous that is, there exists a constant \( c > 0 \) such that

\[ ||v||_{C([0,T];X)} \leq c ||v||_{H^1(0,T;X)}. \]

**Proof** See [8].
**Remark 1.7** For \( v \in H^1(0, T; X) \), \( v(0) \) is understood in the sense of the embedding \( H^1(0, T; X) \hookrightarrow C([0, T]; X) \).

**Theorem 1.4** *(Lebesgue)* See [8] page 118.  
Let \( X \) be a normed space, \( f \in L^1(a,b; X) \). Then

\[
\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} ||f(t) - f(t_0)||_X dt = 0 \quad \text{for almost all } \ t_0 \in (a,b). 
\]  

**(1.28)**

**Remark 1.8** Theorem 1.4 implies that

\[
\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} f(t) dt = f(t_0) \quad \text{for almost all } \ t_0 \in (a,b). 
\]  

**(1.29)**

It follows from the fact that

\[
\lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} (f(t) - f(t_0)) dt ||_X \leq \lim_{h \to 0} \frac{1}{h} \int_{t_0}^{t_0+h} ||f(t) - f(t_0)||_X dt = 0 \quad \text{for almost all } \ t_0 \in (a,b).
\]

### 1.2.2 Minimization of functionals and elliptic variational inequalities

The results of this section will be useful for the proof of the existence and the uniqueness of the solution of the variational problem of elastoplasticity. Before that, we need the following definition.

**Definition 1.11** Let \( H \) be a real Hilbert space, \( a(\cdot, \cdot) : H \times H \to \mathbb{R} \) a bilinear form. \( a(\cdot, \cdot) \) is said to be \( H \)-elliptic if there exists a constant \( c > 0 \) such that \( |a(v, v)| \geq c||v||_H^2 \) for all \( v \in H \).

**Theorem 1.5** Let \( V \) be a Hilbert space, \( a(\cdot, \cdot) : V \times V \to \mathbb{R} \) a continuous, \( V \)-elliptic bilinear form, \( \ell : V \to \mathbb{R} \) a bounded linear functional, and \( j : V \to \mathbb{R} \) a proper, convex, and l.s.c functional on \( V \). There exists a unique \( u \in V \) that satisfies

\[
a(u, v - u) + j(v) - j(u) \geq \ell, v - u > \forall v \in V. 
\]  

**(1.30)**

**Proof:** See [8] page 128.

In general, the primal variational problem of elastoplasticity has the following form: let \( H \) be a Hilbert space and \( K \subset H \) a nonempty closed, convex cone. Find \( w : [0, T] \to H \) such that \( w(0) = 0 \), \( \dot{w}(t) \in K \) for almost all \( t \in (0, T) \) and

\[
a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq \ell(t), z - \dot{w}(t) > \forall z \in K, 
\]  

**(1.31)**

where \( a(\cdot, \cdot) : H \times H \to \mathbb{R} \) is a bilinear form, \( H \)-elliptic in \( H \) or \( K \), \( j : K \to \mathbb{R} \) a nonlinear functional and \( \ell \in H^1(0, T; H') \) is a linear functional.

**AIMS Essay.**

Laurent TCHOUALAG.
Theorem 1.6 (Existence and uniqueness of the solution).
Let $H$ be a Hilbert space; $K \subset H$ a nonempty, closed, convex cone; $a(\cdot, \cdot) : H \times H \to \mathbb{R}$ a bilinear continuous form that is symmetric, bounded, and $H$-elliptic; $\ell \in H^1(0,T;H')$ with $\ell(0) = 0$; and $j : K \to \mathbb{R}$ a nonnegative, convex, positively homogeneous, and Lipschitz continuous functional. Then there exists a unique $w \in H^1(0,T;H)$ solution of the abstract problem (1.31).

Proof
We are going to give the steps of the proof. The details can be found in [8] or [7].
The proof has two steps: firstly, we discretize the time space, and we establish the existence of a family of solutions of the discrete problems. Secondly, we construct a sequence of piecewise linear interpolants of the discrete solutions, and finally we show that this sequence converges in some sense when the time step goes to zero, to a function solution of the continuous problem.

- First step
Let $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, with $t_n - t_{n-1} = k = \frac{T}{N}$.
Since $\ell \in H^1(0,T;H) \hookrightarrow C([0,T],H')$, we can write $\ell_n = \ell(t_n) \ \forall n = 0,1,...,N$ with $\ell_0 = \ell(0) = 0$. Set $w_n = w(t_n)$, $\Delta w_n = w_n - w_{n-1}$ and $\delta w_n = \Delta w_n / k$. From theorem 1.5, there exists a unique sequence $\{w_n\}^N_{n=0}$ with $w_0 = 0$ such that for $n = 1,2,...,N$, $\Delta w_n \in K$ and
\[
a(w_n,z - \Delta w_n) + j(z) - j(\Delta w_n) \geq \langle \ell_n, z - \Delta w_n \rangle \ \forall z \in H. \tag{1.32}
\]
Furthermore, there exists a positive constant $c$ independent of $k$ such that
\[
\max_{1 \leq n \leq N} \|w_n\|_H \leq c \|\ell\|_{L^1(0,T;H')} \tag{1.33}
\]
and
\[
\sum_{n=1}^{N} \|\delta w_n\|_{H}^2 \leq c \|\ell\|_{L^2(0,T;H')}^2. \tag{1.34}
\]

- Second step:
Set $w^k(t) = w_{n-1} + \delta w_n(t - t_{n-1})$ for $t_{n-1} \leq t \leq t_n$, $1 \leq n \leq N$, the piecewise linear interpolation of $\{w_n\}^N_{n=0}$.
For any sequence $\{z_n\}^{N+1}_{n=0} \in H$, with $Z_{N+1} = 0$, define a step function $z(t)$ by
\[
z(t) = z_n \ \text{for} \ t_{n-1} \leq t \leq t_n, \ n = 0,1,...,N-1, \ \text{and} \ n = N.
\]
Divide both side of (1.32) by $k$ obtain
\[
a(w_n,z - \delta w_n) + j(z) - j(\delta w_n) - \langle \ell_n, z - \delta w_n \rangle \geq 0 \ \forall z \in H. \tag{1.35}
\]
From (1.37), show that follows from (1.38) and (1.39) that there exists a subsequence f n g and for any L 2 (0, T; H) it satisfies the variational inequality

\[ \int_0^T [a(w^k(t), z(t) - \dot{w}^k(t)) + j(z(t)) - j(\dot{w}^k(t)) - < \ell(t), z(t) - \dot{w}^k(t) >] dt - \frac{1}{2} k j(z_1) + \frac{1}{2} \epsilon k \int_0^T ||\ell(t)||_H^2 dt \geq 0. \]  

From (1.33) and (1.34), there exits a constant c such that

\[ ||w^k||_{L^\infty(0, T; H)} \leq c \]  

and

\[ ||w^k||_{L^2(0, T; H)} \leq c. \]  

For a fixed time step k_0 > 0, consider the sequence of step-size k_l = 2^{-l} k_0 \ell = 0, 1, 2, \ldots. It follows from (1.38) and (1.39) that there exists a subsequence \{w^{k_i}\} of the sequence w^{k_i} and a function w \in H^1(0, T; H) such that, \{w^{k_i}\} converges weakly * to w (\{w^{k_i}\} \rightharpoonup w) in L^\infty(0, T; H) and \{\dot{w}^{k_i}\} converges weakly to \dot{w} in L^2(0, T; H).

For any t_0 \in (0, T), let h > 0 such that t_0 + h < T and for any z \in K, define

\[ z_0(t) = \begin{cases} z & \text{if } t_0 \leq t \leq t_0 + h \\ \dot{w}(t) & \text{otherwise}, \end{cases} \]

\[ z_0 \in L^2(0, T; K). \]  

Report z_0 into (1.40), obtain

\[ \frac{1}{h} \int_{t_0}^{t_0 + h} [a(w(t), z(t) - \dot{w}(t)) + j(z(t)) - j(\dot{w}(t)) - < \ell(t), z(t) - \dot{w}(t) >] dt \geq 0 \quad \forall z \in K. \]  

since the expression inside the integral belongs to L^1(0, T; H), take the limit when h \to 0 and apply the Lebesgue theorem (theorem 1.4), obtain

\[ a(w(t), z(t) - \dot{w}(t)) + j(z(t)) - j(\dot{w}(t)) - < \ell(t), z(t) - \dot{w}(t) > \geq 0 \quad \text{a.e. } [0, T] \quad \forall z \in K. \]  

Furthermore, w \in L^\infty(0, T; H) and \dot{w} \in L^2(0, T; H) imply that w \in H^1(0, T; H).

\[ f_m \rightharpoonup f \text{ in } X' \iff < f_n, x > \to < f, x >, \text{ for all } x \in X. \]
The uniqueness of the solution follows directly from the $H$-ellipticity of the bilinear form $a(.,.)$.

**Theorem 1.7 (Stability)**

*Under the assumptions of theorem 1.6, the solution of the abstract problem (1.31) depends continuously on $\ell$ i.e for $\ell^{(1)}, \ell^{(2)} \in H^1(0,T;H')$ with $\ell^{(1)}(0) = \ell^{(2)}(0) = 0$, the corresponding solutions $w^{(1)}$ and $w^{(2)}$ satisfy

$$||w^{(1)} - w^{(2)}||_{L^\infty(0,T;H)} \leq c||\ell^{(1)} - \ell^{(2)}||_{L^1(0,T;H')},$$

where $c$ is a positive constant.*

**Proof**

We are going to give an outline of this proof. The details can be found in [8].

Given $\ell^{(1)}, \ell^{(2)} \in H^1(0,T;H)$, with $\ell^{(1)}(0) = \ell^{(2)}(0) = 0$. Let $w^{(1)}$ and $w^{(2)}$ be the corresponding solution. Set $e(t) = w^{(1)} - w^{(2)}$. Taking $w = w^{(1)}$, $z = \dot{w}^{(2)}$ and $\ell = \ell^{(1)}$ in (1.31) and then $w = w^{(2)}$, $z = \dot{w}^{(1)}$ and $\ell = \ell^{(2)}$ in (1.31). Adding the two relations, obtain

$$\frac{1}{2} \frac{d}{dt} a(e(t), c(t)) \leq \langle \ell^{(1)}(t) - \ell^{(2)}(t), \dot{e}(t) \rangle.$$  (1.44)

Integrate (1.44) over $(0,t)$, $t \in [0,T]$ and use the ellipticity of $a(.,.)$ to obtain

$$||w^{(1)} - w^{(2)}||_{L^\infty(0,T;H)} \leq c||\ell^{(1)} - \ell^{(2)}||_{L^1(0,T;H')}.$$  (1.45)
Firstly, in this chapter, we derive the system of equations which describe the elastoplasticity deformation of a material. Secondly, our concern will be to study the well-posedness of this problem i.e. the existence, uniqueness and stability of the solution. The key mathematical results we will use can be found in the monograph by Han and Reddy [8], and [7].

2.1 Initial-boundary value problem for linear elasticity

In this section, our aim is to derive the system of equations which describe the deformation of a linear elastic body. We will focus on the case of “small” deformation.

Let us start with the kinematics study of the deformation.

2.1.1 Strain Tensor

We assume that at $t = 0$, the body occupies the region $\Omega$ called the reference configuration and after a time $t$, it occupies a new region $\Omega_t$ called the current configuration.

Let $x$ be the position of a material point at time $t = 0$, and let $y(x, t)$ be the position of this material point at time $t$. Then $u(x, t) = y(x, t) - x$ represents the displacement. Since the displacement, $u$ alone cannot give us complete information about the deformation of the body, the quantity used to describe the deformation is the strain tensor. This is defined by

$$\eta(u) = \frac{1}{2}(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u).$$

(2.1)

In components, it has the following form

$$\eta_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}).$$
Remark 2.1  
(i) $\eta_{ii}$ gives half of the change in length (squared) in the $i$ direction.

(ii) $\eta_{ij} i \neq j$ gives the measure of the change in angle between two fibers originally in the $i$ and $j$ directions.

(iii) $\eta$ is a symmetric tensor, that is $\eta^T = \eta$.

(iv) When the body undergoes the rigid motion, $\eta(u) = 0$.

Throughout this work, we will assume that the body undergoes infinitesimal deformation, that is the displacement gradient $\nabla u$ is sufficiently small such that the nonlinear term can be neglected. So,

$$\eta(u) = \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).$$  \hspace{1cm} (2.2)

### 2.1.2 Stress tensor; equation of motion

There are two kinds of forces acting on the body

- The body force $f(x, t)$, represents the force per unit reference volume exerted on the material point $x$ at time $t$ by an agent external to the body. Gravity for example.

- Surface traction $S_n(x, t)$.

- **Cauchy stress tensor:** $\sigma$ There exists on $[0, T] \times \Omega$ a second-order tensor field $\sigma$ such that

$$\sigma n = S_n \quad \text{and} \quad \sigma^T = \sigma,$$  \hspace{1cm} (2.3)

for each outward unit vector $n$, [8, Page 25]. In the case of the infinitesimal deformation the equation of motion is given by

$$\text{div}\sigma + f = \rho \ddot{u}.$$  \hspace{1cm} (2.4)
In component form, it is written
\[
\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \ddot{u}_i \quad 1 \leq i \leq 3.
\]

### 2.1.3 Constitutive equations

The relations (2.2) and (2.4) yield nine equations with fifteen unknowns: six for the strain-tensor, six for the stress-tensor and three for displacement (using the fact that \(\varepsilon\) and \(\sigma\) are symmetric). To have a solvable problem, we still need six more equations. Those six equations are derived from the elastic behavior of the material so-called constitutive equation, which is the generalization of Hooke’s law, given by
\[
\sigma = C\varepsilon, \quad (2.5)
\]
where \(C\) is a fourth order tensor called the elastic tensor. It is a linear map from the space of the symmetric second order tensor into itself. \(C\) has the following properties

- Symmetric

\[
C_{ijkl} = C_{jikl} = C_{ijlk} \quad \text{and} \quad C_{ijkl} = C_{klij}, \quad (2.6)
\]

- Invertible. There exists a fourth-order tensor \(A\) called the compliance tensor, such that for any \(\varepsilon\), and for any \(\sigma\) such that \(\varepsilon^T = \varepsilon\), and \(\sigma^T = \sigma\)

\[
A(C\varepsilon) = \varepsilon \quad \text{and} \quad C(A\sigma) = \sigma. \quad (2.7)
\]

**Remark 2.2**

(i) The symmetric property of the tensor \(C\) (2.6) reduces the number of independent components \(C_{ijkl}\) of \(C\) from 81 to 21.

(ii) In the case of isotropic elasticity where the response of the material to a force is independent of its orientation, the components of the elasticity tensor are given by

\[
C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (2.8)
\]
where \(\delta_{ij}\) is the Kronecker symbol. The strain-stress relation is given by

\[
\sigma = \lambda (tr\varepsilon) I + 2\mu \varepsilon, \quad (2.9)
\]
where \(I\) is the identity matrix; \(\lambda\) and \(\mu\) are called Lamé moduli.

### 2.1.4 Initial-boundary value problem for linear elasticity

Let us assume that a linear elastic body initially occupies a domain \(\Omega \subseteq \mathbb{R}^3\), with Lipschitz boundary \(\partial \Omega = \Gamma = \Gamma_D \cup \Gamma_N\), such that \(\Gamma_D = \partial \Omega \setminus \Gamma_N\) and \(\text{meas}(\Gamma_D) > 0\). The system of equations which
describe the deformation of a linear elastic body is given by the equation of motion

$$\text{div}\sigma + f = \rho\ddot{u},$$

(2.10)

the strain-displacement relation

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T),$$

(2.11)

the elastic constitutive relation

$$\sigma = C\varepsilon,$$

(2.12)

the boundary conditions

$$u = \bar{u} \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \sigma n = \bar{s} \quad \text{on} \quad \Gamma_N,$$

(2.13)

the initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad \dot{u}(x, 0) = v_0(x), \quad \text{on} \quad \Omega$$

(2.14)

where $\bar{u}$, $\bar{s}$, $u_0$ and $v_0$ are the given functions.

Figure 2.2: Deformed beam with fixed boundary $\Gamma_D$ and stressed boundary $\Gamma_N$.

2.2 Initial-boundary value problem for elastoplasticity

When a body is deformed, it will initially show an elastic behavior as discussed in the previous section. After a period of time, the body will show another behavior called plastic deformation, characterized by the irreversibility of the deformation. In this section we are going to formulate the primal problem of elastoplasticity (the combination of elasticity and plasticity). We will develop the theory in the case of infinitesimal strain, quasi-static (i.e. the deformation takes place sufficiently slowly such that the inertial term in the equation of motion is neglected), and rate-independent (i.e. the response of the material does not depend on the rate at which the process takes place). We
will also assume that the process is isothermal.

2.2.1 Elastoplastic variables

The set of variables that is used to formulate the problem are

- the displacement \( u \),
- the strain tensor \( \varepsilon \), which is split in two parts: the elastic strain \( e \), due to the elastic behavior, and the plastic strain \( p \), which characterizes the irreversible part of the deformation. Therefore

\[
\varepsilon = e + p.
\]  

(2.15)

- The internal variables \( \xi = (\xi_i)_{i=1}^m \), which characterize the internal restructuring that takes place during plastic behavior, such as hardening. They can be either scalars or tensors.
- The stress tensor \( \sigma \).
- The internal forces \( \chi = (\chi_i)_{i=1}^m \) that are generated as a result of the internal restructuring that occurs during plastic deformation. They are conjugated to internal variables in the same way that the stress is conjugated to the strain. For convenience, we will label \( \Sigma = (\sigma, \chi) \) the generalized stress, and \( P = (p, \xi) \) the generalized strain. Therefore \( \Sigma \) and \( P \) are conjugate in the sense that the product

\[
\Sigma : \dot{P} \equiv \sigma : \dot{\rho} + \chi_i : \dot{\xi}_i,
\]  

(2.16)

represents the rate of dissipation due to the plastic deformation.

2.2.2 Flow law

The plastic behavior of the material is described here within the framework of convex analysis. Let

\[
K = \{ \tau : \phi(\tau) \leq 0 \},
\]  

(2.17)

be the region of admissible stresses, which is assumed to be a closed, convex and nonempty set. The function \( \phi \) is called the yield function. The elastic behavior takes place while \( \Sigma \) lies in the interior of \( K \), (i.e \( \phi(\Sigma) < 0 \)); whereas, the plastic behavior takes place when \( \Sigma \) lies on the boundary of \( K \), (i.e \( \phi(\Sigma) = 0 \)), called the yield surface. Let \( X \) be the set of generalized plastic rate \( \dot{P} \), and let \( X' \) be the set of generalized stresses. Given a generalized stress \( \Sigma \in \partial K \) (boundary of \( K \)), and the associated generalized strain rate \( \dot{P} \); it follows from the maximum plastic work,

\[
(\tau - \Sigma) : \dot{P} \leq 0 \quad \forall \tau \in K \iff \dot{P} \in N_K(\Sigma).
\]  

(2.18)
This relation is called the normality law. Let $D = \sigma_K$, be the support function of $K$. $D$ is also called the dissipation function.

$$D(\dot{P}) = \sup_{\tau \in K} \{\tau : \dot{P}\} = \Sigma : \dot{P}. \quad (2.19)$$

$D$ is conjugate to the indicator function of $K$ and has the following properties, it is \textit{convex, positively homogeneous, l.s.c}, $D(\dot{P}) \geq 0$ and $D(0) = 0$ (i.e a gauge). Therefore, from theorem 1.1 $\dot{P} \in N_K(\Sigma) = \partial D^*(\Sigma) \iff \Sigma \in \partial D(\dot{P})$. This yields two equivalent formulations of the flow law

- the primal formulation

$$\Sigma \in \partial D(\dot{P}) \quad \text{that is} \quad D(Q) - D(\dot{P}) \geq \Sigma : (Q - \dot{P}) \quad \forall \dot{P} \in \text{dom}(D) \quad \forall Q \in X, \quad (2.20)$$

- and the dual formulation

$$\dot{P} \in N_K(\Sigma) \quad \text{that is} \quad (\tau - \Sigma) : \dot{P} \leq 0 \quad \forall \tau \in X'. \quad (2.21)$$

**Remark 2.3**

(i) The assumptions of the convexity of the region of admissible stresses, and the normality law are based on experiment (See [12]).

(ii) If $\phi$ is smooth, $N_K = \nabla \phi(\Sigma)$, then $\exists \lambda \geq 0$ such that

$$\dot{P} = \lambda \frac{\partial \phi(\Sigma)}{\partial \Sigma}. \quad (2.22)$$

$\lambda$ is called plastic multiplier.

### 2.2.3 Primal Initial-boundary value problem for elastoplasticity

We are now going to summarize the system of equations that describe the elastoplastic behavior of a material. We will focus on the case of linear kinematic and linear isotropic hardening. Let $\Omega \subset \mathbb{R}^3$ represents the reference configuration of an elastoplastic body with Lipschitz boundary $\partial \Omega = \Gamma = \Gamma_D \cup \Gamma_N$, such that $\Gamma_D \cap \Gamma_N = \emptyset$ and $\text{meas}(\Gamma_D) > 0$. The unknowns are

- the displacement $u$,
- the plastic strain $\varepsilon$,
- and the internal hardening variables $\xi$, which satisfying the following system of equations

The equilibrium equation

$$\text{div}\sigma + f = 0. \quad (2.23)$$
The additive decomposition of the strain
\[ \varepsilon = e + p. \] (2.24)

The plastic strain is assumed to be incompressible that is
\[ tr(p) = 0. \] (2.25)

The strain displacement relation
\[ \varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T). \] (2.26)

The boundary conditions
\[ u = 0 \quad \text{on} \quad \Gamma_D, \] (2.27)
\[ \sigma n = g \quad \text{on} \quad \Gamma_N, \] (2.28)

where \( g \) is a given function. The initial condition
\[ u(x,0) = 0 \quad \text{on} \quad \Omega. \] (2.29)

The elastic constitutive equation
\[ \sigma = Ce = C(\varepsilon(u) - p). \] (2.30)

The thermodynamic force conjugate to the internal hardening variables \( \xi \)
\[ \chi = -H\xi, \] (2.31)

where \( H \) is a linear operator from \( \mathbb{R}^m \) into itself called the hardening modulus. The flow law is given by \( (\dot{p}, \dot{\xi}) \in K_p = dom(D) \)
\[ D(q, \eta) \geq D(\dot{p}, \dot{\xi}) + \sigma : (q - \dot{p}) + \chi : (\eta - \dot{\xi}) \quad \forall (q, \eta) \in K_p, \] (2.32)

where \( D \) is the support function of \( K \), region of admissible stresses.

### 2.3 Primal variational problem of elastoplasticity

In the previous section, we derived the system of equations which govern the elastoplastic behavior of a material, but it is not easy to study the well-posedness of the problem in this form ((2.23)-(2.32)). We are going to recast a new formulation, called the variational formulation equivalent to the previous one. Before that, we need to make some realistic assumptions on the parameters of the problem that lie on the properties of an elastoplastic material.
2.3.1 Assumptions

(i) The elasticity tensor $C$ has the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{klij}, \quad (2.33)$$

(ii) $C_{ijkl}$ is bounded and measurable that is

$$C_{ijkl} \in L^\infty(\Omega), \quad (2.34)$$

(iii) $C$ is point-wise stable that is there exist positive constant $C_0$ such that

$$C_{ijkl}(x)\varepsilon_{ij}\varepsilon_{kl} \geq C_0|\varepsilon|^2 \quad \text{a.e in } \Omega \quad \forall \varepsilon = (\varepsilon_{ij}) \in M^3. \quad (2.35)$$

(iv) $H$ is symmetric in the sense that

$$\xi : H\lambda = \lambda : H\xi \quad \forall \xi, \lambda \in \mathbb{R}^m, \quad (2.36)$$

(v) $H_{ij}$ is bounded and measurable that is

$$H_{ij} \in L^\infty(\Omega), \quad (2.37)$$

(vi) $H$ is positive definite in the sense that $\exists H_0 > 0$ such that

$$\xi : H\xi \geq H_0|\xi|^2 \quad \text{a.e in } \Omega \quad \forall \xi \in \mathbb{R}^m. \quad (2.38)$$

(vii) $f \in H^1(0,T;[L^2(\Omega)]^3)$ and $f(0,x) = 0$ a.e in $\Omega$. \quad (2.39)

(viii) $g \in H^1(0,T;[L^2(\Gamma)]^3)$ and $g(0,x) = 0$ a.e in $\Gamma$. \quad (2.40)

2.3.2 Function Spaces

Let us define the function spaces in which we are going to find the solution of the problem.

The space $V$ of displacement is defined by

$$V = \{v \in [H^1(\Omega)]^3 : v = 0 \quad \text{a.e on } \Gamma_D\}. \quad (2.41)$$

$V$ is a Hilbert space with the inner product $(u,v)_V = \sum_{i=1}^{3} (u_i,v_i)_{H^1(\Omega)}$ and the associate norm (from
Korn’s inequality, theorem 1.2)
\[ ||v||_V^2 = \int_\Omega |\varepsilon(v)|^2 \, dx. \quad (2.42) \]

Let
\[ Q = \{ q = (q_{ij})_{3 \times 3} : q_{ij} = q_{ji}, \quad q_{ij} \in L^2(\Omega) \}. \quad (2.43) \]

It is a Hilbert space with the usual inner product and norm of the space \([L^2(\Omega)]^{3 \times 3}\). Then the space \( Q_0 \) of plastic strain is the closed subspace of \( Q \) defined by
\[ Q_0 = \{ q \in Q : tr(q) = 0 \; \text{a.e. in} \; \Omega \}, \quad (2.44) \]

which is also a Hilbert space.

The space \( M \) of internal variables is defined by
\[ M = [L^2(\Omega)]^m, \quad \text{is a Hilbert space with the usual inner product.} \quad (2.45) \]

Therefore the space \( Z = V \times Q_0 \times M \) is a Hilbert space with the inner product
\[ (w, z)_Z = (u, v)_V + (p, q)_Q + (\xi, \eta)_M, \quad (2.46) \]

where \( w = (u, p, \xi) \) and \( z = (v, q, \eta) \)

Let
\[ Z_p = \{ z = (v, q, \eta) \in Z : (q, \eta) \in K_p \; \text{a.e. in} \; \Omega \}, \quad (2.47) \]

which is a nonempty, closed, convex cone in \( Z \).

### 2.3.3 Primal variational formulation

To obtain the variational formulation of the problem, let us start by substituting (2.30) and (2.31) in (2.32). Integrating over \( \Omega \), we obtain
\[ \int_\Omega D(q, \eta) \, dx \geq \int_\Omega D(\hat{p}, \hat{\xi}) \, dx + \int_\Omega [C(\varepsilon(u) - p) : (q - \hat{p}) - H\xi : (\eta - \hat{\xi})] \, dx \quad \forall (q, \eta) \in Z_p. \quad (2.48) \]

We substitute (2.30) into (2.23), and take the scalar product with \( v - \hat{u} \) for arbitrary \( v \in V \). Then, we integrate over \( \Omega \) and use the Green formula to obtain
\[ \int_\Omega [C(\varepsilon(u) - p) : (\varepsilon(v) - \varepsilon(\hat{u}))) \, dx = \int_\Omega f(v - \hat{u}) \, dx + \int_{\Gamma_N} g(v - \hat{u}) \, ds \quad \forall v \in V. \quad (2.49) \]

Now adding (2.48) and (2.49), we obtain the following variational inequality
\[ a(w(t), z - \hat{w}(t)) + j(z) - j(\hat{w}(t)) \geq < \ell(t), z - \hat{w}(t) >, \quad (2.50) \]
where \( a(.,.) : Z \times Z \rightarrow \mathbb{R} \) is a bilinear form defined by

\[
a(w, z) = \int_{\Omega} [C(\varepsilon(u) - p) : \varepsilon(v) - q + \xi : H\eta]dx,
\]

(2.51)

and \( \ell(t) : Z \rightarrow \mathbb{R} \) is a linear functional defined by

\[
< \ell(t), z > = \int_{\Omega} f(t)vdx + \int_{\Gamma_N} g(t).vds,
\]

(2.52)

and \( j : Z \rightarrow \mathbb{R} \) is a functional defined by

\[
j(z) = \int_{\Omega} D(q, \eta)dx,
\]

(2.53)

where \( w = (u, p, \xi) \) and \( z = (v, q, \eta) \). Therefore, the primal variational problem of elastoplasticity has the following form. Find \( w = (u, p, \xi) : [0; T] \rightarrow Z \), such that \( w(0) = 0 \), and for almost all \( t \in (0, T) \), \( \dot{w}(t) \in Z_p \) and

\[
a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) \geq < \ell(t), z - \dot{w}(t) > \quad \forall z \in Z_p.
\]

(2.54)

**Proposition 2.1** The initial boundary-value problem (2.23)-(2.32) and the variational problem (2.54) are equivalent.

**Proof**

To establish the equivalence, we just need to prove that the solution of the variational problem (2.54) is also the solution of the initial boundary-value problem (2.23)-(2.32). The other side follows from the construction of the variational (2.54).

Let \( w = (u, p, \xi) \) be the solution of (2.54), \( v \in V \subset [H^1(\Omega)]^3 \) and \( z = (\dot{u} + v, \dot{p}, \dot{\xi}) \in Z \).

By substituting \( w \) and \( z \) in (2.54) we obtain

\[
\int_{\Omega} C(\varepsilon(u) - p) : \varepsilon(v)dx \geq \int_{\Omega} f(t)vdx + \int_{\Gamma_N} g(t).vds, \quad \forall v \in V \subset [H^1(\Omega)]^3.
\]

(2.55)

Since \([H^1(\Omega)]^3\) is a vector space, by substituting \( v \) by \(-v\) in (2.55) we obtain

\[
\int_{\Omega} C(\varepsilon(u) - p) : \varepsilon(v)dx \leq \int_{\Omega} f(t)vdx + \int_{\Gamma_N} g(t).vds, \quad \forall v \in V \subset [H^1(\Omega)]^3,
\]

(2.56)

then from (2.55) and (2.56) we obtain

\[
\int_{\Omega} C(\varepsilon(u) - p) : \varepsilon(v)dx = \int_{\Omega} f(t)vdx + \int_{\Gamma_N} g(t).vds, \quad \forall v \in V \subset [H^1(\Omega)]^3.
\]

(2.57)
In particular for \( v \in (C_0^\infty(\Omega))^3 \subset [H^1(\Omega)]^3 \) we obtain
\[
\int_\Omega C(\varepsilon(u) - p) : \varepsilon(v) \, dx = \int_\Omega f(t) \, v \, dx. \tag{2.58}
\]
Integrate (2.58) by parts we obtain
\[
-\int_\Omega \text{div}(C(\varepsilon(u) - p)) \, v \, dx = \int_\Omega f(t) \, v \, dx, \quad \forall v \in (C_0^\infty(\Omega))^3. \tag{2.59}
\]
Thus
\[
\int_\Omega (\text{div}(C(\varepsilon(u) - p)) + f(t)) \, v \, dx = 0, \quad \forall v \in (C_0^\infty(\Omega))^3,
\]
therefore
\[
\text{div}(C(\varepsilon(u) - p)) + f(t) = 0, \quad \text{a.e. in } \Omega. \tag{2.60}
\]
Since \( w \in Z_p \) and \( w(0) = 0 \), then
\[
u = 0 \quad \text{a.e. on } \Gamma_D \quad \text{and} \quad u(x, 0) = 0 \quad \text{a.e. in } \Omega.
\]
On the other hand, from Green’s formula we have
\[
\int_\Omega \text{div}(C(\varepsilon(u) - p)) \, v \, dx = -\int_\Omega C(\varepsilon(u) - p) : \varepsilon(v) \, dx + \int_{\Gamma_N} \frac{\partial(C(\varepsilon(u) - p))}{\partial n} \, v ds,
\]
\[\forall v \in V \subset [H^1(\Omega)]^3. \tag{2.62}\]
From (2.57) and (2.62) we deduce that
\[
\int_{\Gamma_N} \left( \frac{\partial(C(\varepsilon(u) - p))}{\partial n} - g(t) \right) \, v ds = 0, \quad \forall v \in V \subset [H^1(\Omega)]^3,
\]
thus,
\[\sigma n = g \quad \text{a.e. on } \Gamma_N \quad \text{with} \quad \sigma = C(\varepsilon(u) - p). \tag{2.63}\]
To obtain the flow law, replace \( v \) in (2.57) by \( v - \dot{u} \) and substitute in (2.54) we obtain
\[
\int_\Omega (\sigma : (q - \dot{p}) - H \xi : (\eta - \dot{\xi})) \, dx + j(z) - j(\dot{w}) \geq 0, \tag{2.65}
\]
then,
\[
\int_\Omega (\sigma : (q - \dot{p}) + \chi : (\eta - \dot{\xi}) + D(q, \eta) - D(\dot{p}, \dot{\xi})) \, dx \geq 0, \tag{2.66}
\]
therefore,
\[D(q, \eta) - D(\dot{p}, \dot{\xi}) \geq \sigma : (q - \dot{p}) + \chi : (\eta - \dot{\xi}). \tag{2.67}\]

**Remark 2.4** The primal variational problem for linear elasticity can be deduced from (2.54) by taking \( p = 0 \) and \( \xi = 0 \).
2.3.4 Analysis of the primal variational problem

We have reached the point at which we can now study the well-posedness of the primal problem of elastoplasticity. The mathematical theory of such a problem can be found in [1], [8] and [6]. We will focus on the case of plasticity with combined isotropic and kinematic hardening. In this case, there are two internal variables:

- The plastic strain $p$, which is the kinematic-type internal variable. The corresponding conjugate force is denoted by $a$.
- The accumulated plastic strain $\gamma$, a positive scalar which is the isotropic-type internal variable. The corresponding scalar conjugate force is denoted by $G$.

We introduce the domain of admissible stresses (or elastic domain)

$$K = \{(\sigma, a, G) \in \mathbb{R}^{3\times3}_{\text{sym}} \times \mathbb{R}^{3\times3} \times \mathbb{R} : \phi(\sigma, a, G) \leq 0, G \leq 0\},$$  \hspace{1cm} (2.68)

where $\phi(\sigma, a, G)$ is the continuous and convex Von-Mises yield function

$$\phi(\sigma, a, G) = |\text{dev}(\sigma + a)| + G - c_0$$  \hspace{1cm} (2.69)

and $c_0$ is the material constant referred to as the yield stress. The support function of $K$ is then defined as follows

$$D(q, \mu) = \begin{cases} c_0|q| & \text{if } |q| \leq \eta \text{ and } \text{tr}q = 0 \\ \infty & \text{if } |q| > \eta. \end{cases}$$  \hspace{1cm} (2.70)

The domain of $D$ in this case is

$$K_p = \{(q, \eta) \in Q_0 \times M : |q| \leq \eta\},$$  \hspace{1cm} (2.71)

and the domain in which we are going to find the solution is

$$Z_p = \{(v, q, \eta) \in Z : (q, \eta) \in K_p\}. \hspace{1cm} (2.72)$$

We assume a linear hardening law in the form

$$a = -\mathbb{H}p \quad \text{and} \quad G = -H\gamma,$$  \hspace{1cm} (2.73)

with $\mathbb{H}$ symmetric, positive definite and such that

$$\mathbb{H}\delta : \delta \geq h_1|\delta|^2, \quad h_1 \quad \text{is a positive constant}$$  \hspace{1cm} (2.74)
and $H$ the isotropic hardening modulus. We set

$$h_2 = \min_{x \in \Omega} H(x) > 0.$$  \hspace{1cm} (2.75)

(2.74) and (2.75) are the reformulations of hypothesis (Vi) (2.38).

**Theorem 2.1** Under assumptions (2.33) - (2.37), (2.39),(2.40), (2.74), (2.75), the primal variational problem of elastoplasticity (2.54) has a unique solution, and the solution depends continuously on $f$ and $g$.

**Proof**

- **Existence and uniqueness of the solution.** Let us first verify the $Z$-ellipticity of the bilinear form $a(\cdot;\cdot)$ defined by (2.51). Let $z = (v,q,\eta) \in Z$, from (2.35), (2.74), (2.75) and from Young’s inequality, we have

$$a(z,z) \geq C_0 \int \Omega |\varepsilon(v) - q|^2 dx + h_1 \int \Omega |q|^2 dx + h_2 \int \Omega |\eta|^2 dx,$$

$$\geq C_0 \int \Omega |\varepsilon(v)|^2 - 2\varepsilon(v) : q + |q|^2 |dx + h_1 \int \Omega |q|^2 dx + h_2 \int \Omega |\eta|^2 dx,$$

$$\geq C_0 \int \Omega |\varepsilon(v)|^2 - \theta^{-1}|q|^2 - \theta |\varepsilon(v)|^2 + |q|^2 |dx + h_1 \int \Omega |q|^2 dx + h_2 \int \Omega |\eta|^2 dx,$$

$$\geq C_0(1 - \theta) \int \Omega |\varepsilon(v)|^2 dx + |C_0(1 - \theta^{-1}) + h_1| \int \Omega |q|^2 dx + h_2 \int \Omega |\eta|^2 dx,$$

for every $\theta \in (0,1)$.

Choosing $\theta \in \left(\frac{C_0}{C_0 + h_1} : 1\right)$ and $\alpha = \min\{C_0(1 - \theta) ; C_0(1 - \theta^{-1}) + h_1 ; h_2\} > 0$ we obtain

$$a(z,z) \geq \alpha \left( \int \Omega |\varepsilon(v)|^2 dx + \int \Omega |q|^2 dx + \int \Omega |\eta|^2 dx \right),$$

for all $z = (v,q,\eta) \in Z_p$, from which Korn’s inequality yields

$$a(z,z) \geq \alpha ||z||^2_Z \quad \text{for all} \quad z \in Z_p.$$

Thus $a(\cdot,\cdot)$ is $Z$-elliptic.

It is obvious that $a(\cdot,\cdot)$ is continuous, bounded and symmetric. In addition the functional $j(.)$ defined by (2.53) is nonnegative, convex, positively homogeneous, and is Lipschitz continuous on $Z_p$ this follows from the definition of $D$. We can also easily check that $\ell(.)$ defined by (2.52) belongs to $H^1(0,T;Z')$ and $\ell(0) = 0$. Therefore from theorem 1.6, with $H = Z$ and $K = Z_p$, we can deduce that the primal variational problem of elastoplasticity with combined kinematic and isotropic linear hardening has a unique solution $w = (u,p,\xi) \in H^1(0,T;Z)$.
**Stability**

Let \( f_1, f_2 \in H^1(0,T; [L^2(\Omega)]^3) \) and \( g_1, g_2 \in H^1(0,T; [L^2(\Gamma)]^3) \) with \( f_1(x,0) = f_2(x,0) = g_1(x,0) = g_2(x,0) = 0 \). Let \( \ell_i, i = 1, 2 \) be defined by

\[
\langle \ell_i(t), z \rangle = \int_{\Omega} f_i(t)vdx + \int_{\Gamma_N} g_i(t)vd\nu,
\]

and denote by \( w_i, i = 1, 2 \) the corresponding solution. From theorem 1.7 we have

\[
||w_1 - w_2||_{L^\infty(0,T;Z)} \leq c ||\ell_1 - \ell_2||_{L^1(0,T;Z')},
\]

where \( c \) is a positive constant independent of \( t \). Thus the solution depends continuously on \( f \) and \( g \).
EXTENDED MODEL OF ELASTOPLASTICITY

In this chapter we will explore the concepts and details needed for constructing the flow law. Firstly, for a given yield function, we will find the dissipation function and establish its well-posedness. Secondly, those results are used to prove both the existence and uniqueness for the solution of the primal variational problem. We mainly focus on the following yield functions, obtained by some engineer’s experiment [10].

3.1 The yield functions

The interest in this work lies in the typical yield surfaces with high curvature,

\[ f(\sigma, \chi) = \frac{\sigma^{(m-1)/2m}}{\sqrt{3}} \left( \sum_{i=1}^{3} [\sigma_i - [\sigma]_{mod(i+1,3)}]^{2m} \right)^{1/2m} - \sqrt{\frac{2}{3}} (\sigma_0 - \chi), \quad 1 \leq m \leq \infty, \quad \text{with} \quad \sigma_0 \geq 0 \quad \text{and} \quad \chi \leq 0, \]

(3.1)

where \( \sigma_0 \) is the initial yield limit, \( \chi \) the internal stress for the considered isotropic strain hardening and \( m \) the material parameter which determines the shape of the yield surface.

Throughout this work, most of our computation will be done in the deviatoric plane. So it might be helpful to give a short review about how these surfaces can be represented in the stress space.

3.1.1 Geometric representation of stress spaces

Using the three principal stresses \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) as the coordinates, a three dimensional stress space can be constructed. This stress representation is known as the Haigh-Westergaard stress space [9], and is shown in figure 3.1 (Appendix). In the following, we assume the principal stresses are ordered according to \( \sigma_3 \leq \sigma_2 \leq \sigma_1 \). The decomposition of a stress state into a spheric (hydrostatic) denoted...
by
\[ S = \frac{1}{3} \text{tr}(\sigma) I \]  
(3.2)
and deviatoric given by
\[ \text{dev}(\sigma) = \sigma^D = \sigma - \frac{1}{3} \text{tr}(\sigma) I, \]  
(3.3)
where \( \text{tr}(\sigma) \) represents the trace of \( \sigma \) and \( I \) the identity matrix. Stress components can be geometrically represented in this space.

Considering an arbitrary stress state \( OP \) starting from \( O(0,0,0) \) and ending at \( P(\sigma_1, \sigma_2, \sigma_3) \). The vector \( OP \) can be decomposed into two components \( ON \) and \( NP \), where \( ON \) is along the direction of the unit vector \( \frac{1}{\sqrt{3}}(1,1,1) \) and \( NP \) is orthogonal to \( ON \). The vector \( ON \) represents the spheric (hydrostatic) component of the stress state. The axis \( OJ \) is called hydrostatic axis, and every point on this axis has \( \sigma_1 = \sigma_2 = \sigma_3 \). The vector \( NP \) represents the deviatoric component of the stress state \( (\sigma_1^D, \sigma_2^D, \sigma_3^D) \) and is perpendicular to the axis \( OJ \). Any plane perpendicular to the hydrostatic axis is called the deviatoric plane is expressed as
\[ \sigma_1 + \sigma_2 + \sigma_3 = \text{cst.} \]  
(3.4)

**Remark 3.1**
- The particular deviatoric plane which passes through the origin is called the \( \pi \)-plane \([9]\) and is represented by
\[ \sigma_1 + \sigma_2 + \sigma_3 = 0. \]  
(3.5)
- Any plane containing the hydrostatic axis is called a meridian plane \([9]\). The vector \( NP \) lies in the a meridian plane and has
\[ ||NP|| = \sqrt{(\sigma_1^D)^2 + (\sigma_2^D)^2 + (\sigma_3^D)^2}. \]  
(3.6)

### 3.2 Dissipation function

Throughout this work, our analysis will be done by using the three principal stresses as coordinates, and in the \( \pi \)-plane. Therefore the internal variables \((p, \xi)\) can be found in \( Q \times \mathbb{R} \), where
\[ Q = \{ p \in \mathbb{R}^3 : p_1 + p_2 + p_3 = 0 \quad \text{i.e.} \quad \text{tr}(p) = 0 \} \]
and \( \xi \) denotes the hardening parameter.

The space of admissible stresses is defined by
\[ \mathbb{K}_{2m} = \{ (\sigma, \chi) \in Q' \times \mathbb{R} : \chi \leq 0, f(\sigma, \chi) \leq 0 \}, \]
where
\[ Q' = \{ \sigma \in \mathbb{R}^3 : \sigma_1 + \sigma_2 + \sigma_3 = 0 \quad \text{i.e.} \quad \text{tr}(\sigma) = 0 \} \]
is the dual space of $\mathbb{Q}$.

**Remark 3.2**  
- The duality on $\mathbb{Q}$ and $\mathbb{Q}'$ is defined by $\forall (\sigma, q) \in \mathbb{Q}' \times \mathbb{Q}$,
  
  \[
  \sigma : q = \sigma^D : q \quad \text{because} \quad \text{tr}(q) = 0, \\
  = \sigma^D : q^D \quad \text{because} \quad \text{tr}(\sigma^D) = 0.
  \]

Therefore we can define on $\mathbb{Q}$ a new inner product by

\[
\forall (p, q) \in \mathbb{Q} \times \mathbb{R}, \quad p : q = (p_1 - p_2)(q_1 - q_2) + (p_2 - p_3)(q_2 - q_3) + (p_3 - p_1)(q_3 - q_1). \tag{3.7}
\]

- The same inner product can be defined on $\mathbb{Q}'$. Furthermore its associated norm is given by

\[
\forall \sigma \in \mathbb{Q}' \quad |\sigma|_{0,2} = \left( \sum_{i=1}^{3} \left( |\sigma|_i - |\sigma|_{\text{mod}(i+1,3)} \right)^2 \right)^{1/2}. \tag{3.8}
\]

**Proposition 3.1** For $1 \leq m < \infty$,

\[
|\sigma|_{0,2m} = \left( \sum_{i=1}^{3} \left( |\sigma|_i - |\sigma|_{\text{mod}(i+1,3)} \right)^{2m} \right)^{1/2m}, \tag{3.9}
\]

is a norm in the $\pi$-plane. Furthermore, it is equivalent to the canonical norm in the $\pi$-plane defined by

\[
|\sigma|_{2m} = \left( \sum_{i=1}^{3} (\sigma^D_i)^{2m} \right)^{1/2m}. \tag{3.10}
\]

**Proof**

For $m = 1, 2$ we have $|\sigma|_{2m} = \sqrt[3]{3} |\sigma|_{0,2m}$ for all $\sigma \in \mathbb{Q}'$.

On other hand since $\sigma_3 \leq \sigma_2 \leq \sigma_1$, for $3 \leq m < \infty$ we have

\[
\frac{2^{1/2m}}{3} |\sigma|_{0,2m} \leq |\sigma|_{2m} \leq 2 \times \frac{2^{1/2m}}{3} |\sigma|_{0,2m} \quad \forall \sigma \in \mathbb{Q}'. \tag{3.11}
\]

**Lemma 3.1** for $1 \leq r \leq \infty$,

\[
\forall \sigma \in \mathbb{Q}', p \in \mathbb{Q}, \quad |\sigma : p| \leq |\sigma|_{0,r} |p|_{0,s} \quad \text{with} \quad \frac{1}{r} + \frac{1}{s} = 1,
\]

where $|.|_{0,r}$ and $|.|_{0,s}$ are respectively the norms on $\mathbb{Q}'$ and $\mathbb{Q}$.

**Proof** (see Appendix).
Proposition 3.2 Given an admissible stresses domain

\[ K_r = \{ \tau \in \mathbb{R}^n : |\tau|_{0,r} \leq 1 \}, \]

where \( |x|_{0,r} = \left( \sum_{i=1}^{3} [x]_i \right)^{1/r} \), with \( 1 \leq r \leq \infty \). The support function of \( K_r \) is

\[ D_r(q) = |q|_{0,s}, \quad \text{where} \quad \frac{1}{r} + \frac{1}{s} = 1. \quad (3.12) \]

Proof

- For \( 1 < r < \infty \),

\[ D_r(q) = \sup_{\sigma \in K_r} \sigma : q \leq \sup_{\sigma \in K_r} |\sigma : q|, \]

\[ \leq \sup_{\sigma \in K_r} |\sigma|_{0,r} |q|_{0,s} \quad \text{with} \quad \frac{1}{r} + \frac{1}{s} = 1 \quad (\text{lemma 3.1}) , \]

where \( D_r(q) \leq |q|_{0,s} \).

Since, \( (\mathbb{R}^n)' = \mathbb{R}^n \), take \( \sigma = \frac{1}{|q|_{0,s}} (|q|_{s-2} q_1, |q|_{s-2} q_2, ..., |q|_{s-2} q_n) \), we have \( |\sigma|_{0,r} = 1 \), then \( \sigma \in K_r \), and on the other hand \( \sigma : q = |q|_{0,s} \leq \sup_{\tau \in K_r} \tau : q = D_r(q) \).

Therefore \( D_r(q) = \sup_{\sigma \in K_r} \sigma : q = |q|_{0,s} \) with \( \frac{1}{r} + \frac{1}{s} = 1 \).

- For \( r = \infty \),

\[ D_\infty(q) = \sup_{\sigma \in K_\infty} \sigma : q \leq \sup_{\sigma \in K_\infty} |\sigma|_{0,\infty} |q|_{0,1} \quad (\text{lemma 3.1}) , \]

\[ \leq |q|_{0,1}, \]

so, \( D_\infty(q) \leq |q|_{0,1} \).

On the other hand, let \( \sigma = (\epsilon_1, \epsilon_2, ..., \epsilon_n) \), where

\[ \epsilon_i = \begin{cases} -1 & \text{if } q_i \leq 0 \\ 1 & \text{if } q_i > 0, \end{cases} \]

therefore, \( \sigma \in K_\infty \) because \( |\sigma|_{0,\infty} = 1 \), furthermore, \( \sigma : q = |q|_{0,1} \leq \sup_{\tau \in K_\infty} \tau : q = D_\infty(q) \).

Thus, \( D_\infty(q) = |q|_{0,1} \).
3.2.1 The principle of the maximum dissipation

For our analysis we consider the yield function,

\[
f(\sigma, \chi) = \frac{2^{(m-1)/2m}}{\sqrt{3}} \left( \sum_{i=1}^{3} (|\sigma|_i - |\sigma|_{mod(i+1,3)})^{2^{m}} \right)^{1/2m} - \sqrt{\frac{2}{3}}(\sigma_0 - \chi), \quad 1 \leq m \leq \infty, \quad \text{with } \sigma_0 \geq 0 \text{ and } \chi \leq 0,
\]

where \(\sigma_0\) is the initial yield limit, \(\chi\) the internal stress for the considered isotropic strain hardening and \(m\) the material parameter.

We set \(\Sigma = (\sigma, \chi)\) the generalized stress.

The principle of maximum plastic dissipation states that, for given plastic strain \(P\), among all possible stresses \(T\) satisfying the yield criterion, the plastic dissipation attains its maximum for the actual stress tensor \(\Sigma\), that is let \(K_{2m}\) be the closure of the elastic range in stress space, which we recall as

\[
K_{2m} = \{ \Sigma = (\sigma, \chi) \in \mathbb{Q} \times \mathbb{R} : f(\Sigma) \leq 0, \quad \chi \leq 0 \}.
\]

Let seek the dissipation function for each value of \(m \in \mathbb{Z}^+ \setminus \{0\}\), all results in this section will follow from the proposition 3.2.

Dissipation functions.

- For \(m = 1\),
  the yield function is defined by

\[
f(\sigma, \chi) = \frac{1}{\sqrt{3}}|\sigma|_{0,2} - \sqrt{\frac{2}{3}}(\sigma_0 - \chi), \quad \text{ (3.14)}
\]

where \(\sigma_0 \geq 0\) and \(\chi \leq 0\). Then the elastic domain is

\[
K_2 = \{ (\sigma, \chi) \in \mathbb{Q} \times \mathbb{R} : |\sigma|_{0,2} \leq \sqrt{2}(\sigma_0 - \chi), \quad \sigma_0 \geq 0, \quad \chi \leq 0 \}.
\]

The dissipation function is, \(\forall (q, \mu) \in \mathbb{Q} \times \mathbb{R}, \quad \mu \geq 0\)

\[
D_2(q, \mu) = \sup_{(\sigma, \chi) \in K_2} (\sigma : q + \chi \mu),
\]

\[
\leq \sup_{|\sigma|_{0,2} \leq \sqrt{2}(\sigma_0 - \chi)} (|\sigma|_{0,2}|q|_{0,2} + \chi \mu), \quad \text{(lemma 3.1)},
\]

\[
\leq \sup_{\chi \leq 0} \left( \sqrt{2}\sigma_0|q|_{0,2} + (\mu - \sqrt{2}|q|_{0,2})\chi \right),
\]

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where from proposition 3.2 the dissipation function is given by

\[
D_2(q, \mu) = \begin{cases}
\sqrt{2\sigma_0} |q|_{0,2} & \text{if } |q|_{0,2} \leq \frac{1}{\sqrt{2}} \mu \\
\infty & \text{if } |q|_{0,2} > \frac{1}{\sqrt{2}} \mu,
\end{cases}
\tag{3.15}
\]

where \(|x|_{0,2}^2 = \sum_{i=1}^{3} |x_i - x_{mod(i+1,3)}|^2\) is a norm on \(Q \subset \mathbb{R}^3\).

- For \(m \to \infty\),

**Lemma 3.2**

\[
\lim_{m \to \infty} |\sigma|_{0,2m} = \max_{i,j \in \{1,2,3\}} |\sigma_i - \sigma_j| = |\sigma|_{0,\infty}.
\tag{3.16}
\]

**Proof**

\[
\max_{i,j} |\sigma_i - \sigma_j| \leq |\sigma|_{0,2m} \leq 3^{1/2m} \max_{i,j} |\sigma_i - \sigma_j|,
\tag{3.17}
\]

take the limit when \(m \to \infty\) we obtain the result.

From this result, the yield function is given by

\[
f(\sigma, \chi) = \sqrt{\frac{2}{3}} |\sigma|_{0,\infty} - \sqrt{\frac{2}{3}} (\sigma_0 - \chi), \quad \text{with } \sigma_0 \geq 0 \quad \text{and} \quad \chi \leq 0.
\tag{3.18}
\]

Therefore, the admissible domain is

\[
K_{\infty} = \{ \Sigma = (\sigma, \chi) \in Q' \times \mathbb{R} : |\sigma|_{0,\infty} \leq (\sigma_0 - \chi), \quad \sigma_0 \geq 0, \quad \chi \leq 0 \};
\]

Furthermore, the dissipation function is given by

\[
D_\infty(q, \mu) = \begin{cases}
\sigma_0 |q|_{0,1} & \text{if } |q|_{0,1} \leq \mu \\
\infty & \text{if } |q|_{0,1} > \mu,
\end{cases}
\tag{3.19}
\]

where \(|x|_{0,1} = \sum_{i=1}^{3} |x_i - x_{mod(i+1,3)}|\) is a norm on \(Q \subset \mathbb{R}^3\).

**Remark 3.3** *The yield surface \(f = 0\) in the stress deviatoric plane (\(\pi\)-plane), recovering the Von-Mises circle for \(m = 1\) and the Tresca hexagon for \(m \to \infty\) see figure 3.2 (see Appendix).*

- For \(1 < m < \infty\), the dissipation function is given by the following proposition.
Proposition 3.3 for $1 < m < \infty$, the elastic region is

$$K_{2m} = \{ \Sigma = (\sigma, \chi) \in \mathbb{Q}' \times \mathbb{R} : |\sigma|_{0,2m} \leq \sqrt{\frac{2}{C_m}}(\sigma_0 - \chi), \quad \sigma_0 \geq 0, \quad \chi \leq 0 \},$$

where $C_m = \frac{2^{(m-1)/2m}}{\sqrt{3}}$. Therefore, the dissipation function is given by

$$\forall (q, \mu) \in \mathbb{Q} \times \mathbb{R}^+ \quad D_{2m}(q, \mu) = \begin{cases} \frac{\sqrt{2}}{C_m} \sigma_0 |q|_{0,r} & \text{if } |q|_{0,r} \leq \sqrt{\frac{2}{C_m}} \mu, \\ \infty & \text{if } |q|_{0,r} > \sqrt{\frac{2}{C_m}} \mu, \end{cases} \quad (3.20)$$

where $|x|_{0,r} = \left( \sum_{i=1}^{3} |x_i - x_{\text{mod}(i+1,3)}|^r \right)^{1/r}$ is a norm on $\mathbb{Q} \subset \mathbb{R}^3$.

Proof (see Appendix).

Let us now applying these results to the primal problem of elastoplasticity with isotropic hardening.

3.3 Applications

In this section, since the case of kinematic and combined kinematic and isotropic hardening are obvious, we will focus in the case of isotropic hardening.

3.3.1 The constitutive Law

Let $\Sigma = (\sigma, \chi)$ be the generalized stress, and let $P = (p, \xi)$ be the generalized strain tensor. The constitutive law is given by

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T), \quad (3.21)$$

$$\varepsilon = e + p, \quad (3.22)$$

$$\text{tr}(p) = 0, \quad (3.23)$$

$$\sigma = \mathbb{C}e = \mathbb{C}(\varepsilon(u) - p), \quad (3.24)$$

$$\mathbb{C}e = 2\mu e + \lambda \text{tr}(e)I, \quad (3.25)$$

$$\chi = -k\xi, \quad (3.26)$$

$$\Sigma = (\sigma, \chi) \in K_{2m}, \quad \dot{\tau} : (\tau - \sigma) + \dot{\xi} : (\gamma - \chi) \leq 0 \quad \text{for all} \quad T = (\tau, \gamma) \in K_{2m}, \quad (3.27)$$
where $\mu$ and $\lambda$ are the positive coefficients called Lamé moduli, $k$ is a positive coefficient of isotropic hardening and

$$K_{2m} = \{ \Sigma = (\sigma, \chi) \in Q' \times R : |\sigma|_{0,2m} \leq \sqrt[3]{2} C_m (\sigma_0 - \chi), \quad \sigma_0 \geq 0, \quad \chi \leq 0 \},$$

where $C_m = \frac{\sqrt{(m-1)/2m}}{\sqrt{3}}$. We define, the canonical norm and the corresponding scalar product

$$|a|^2 = a : a; \quad a : b = \sum_{i,j=1}^{3} a_{ij} b_{ij}. \quad (3.28)$$

The following lemma reformulates the variational inequality (3.27) in term of the dissipation function $D_{2m}$, (see [2], page 3).

**Lemma 3.3** Let $\hat{P} = (\hat{p}, \hat{\xi}) \in Q \times R$, $\Sigma = (\sigma, \chi) \in Q' \times R$. Then

$$\Sigma = (\sigma, \chi) \in K_{2m}, \quad \hat{p} : (\tau - \sigma) + \hat{\xi} : (\gamma - \chi) \leq 0 \quad \text{for all} \quad T = (\tau, \gamma) \in K_{2m}, \quad (3.29)$$

hold if and only if

$$\sigma : (q - \hat{p}) + \chi : (\eta - \hat{\xi}) \leq D_{2m}(q, \eta) - D_{2m}(\hat{p}, \hat{\xi}) \quad \forall (q, \eta) \in Q \times R, \quad (3.30)$$

where, $D_{2m} : Q \times R \rightarrow R \cup \{\infty\}$

$$D_{2m}(q, \eta) = \begin{cases} \sqrt[3]{2} C_m |\sigma|_{0,r} & \text{if} \quad |\sigma|_{0,r} \leq \sqrt[3]{2} C_m \eta \\ \infty & \text{if} \quad |\sigma|_{0,r} > \sqrt[3]{2} C_m \eta, \quad \text{with} \quad \frac{1}{2m} + \frac{1}{r} = 1 \quad \text{and} \quad \eta \geq 0. \end{cases} \quad (3.31)$$

Where, $|x|_{0,r} = \left( \sum_{i=1}^{3} |x_i - x_{mod(i+1,3)}|^r \right)^{1/r}$ is a norm on $Q \subset R^3$, $C_m = \frac{\sqrt{(m-1)/2m}}{\sqrt{3}}$ and $r = \frac{2m}{2m-1}$.

**Proof** see [2], page 3.

### 3.3.2 The boundary value problem

The elastoplastic continuum is assumed to occupy a bounded domain $\Omega \subset R^3$, with a Lipschitz boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where $\Gamma_D$ is a Dirichlet boundary with $meas(\Gamma_D) > 0$ and $\Gamma_N$ the Neumann part.

The equilibrium equation is given by

$$\text{div}(\sigma(x,t)) + f(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T). \quad (3.32)$$
The boundary conditions are

\[ u = 0 \quad \text{on} \quad \Gamma_D \quad \text{and} \quad \sigma n = g \quad \text{on} \quad \Gamma_N. \]  

(3.33)

The initial condition

\[ u(x, 0) = 0 \quad \text{on} \quad \Omega. \]  

(3.34)

where \( g \) is a given applied surface force, \( n \) denotes the outward normal to the boundary \( \Gamma_N \) and \( f \) is the body force per unit volume. We assume

\[ f(x, 0) = g(x, 0) = 0 \quad \text{on} \quad \Omega. \]  

(3.35)

Our analysis will be restricted to the following functional spaces

- the displacement \( u \) is found in

\[ \mathcal{V} = \{ v \in (H^1(\Omega))^3 : v = 0 \quad \text{on} \quad \Gamma_D \}, \]

- the plastic strain \( p \) is found in

\[ Q_0 = \{ q : q_{ij} = q_{ji} \in L^2(\Omega) \quad \text{and} \quad tr(q) = 0 \}, \]

- the internal variable \( \xi \) will be in \( L^2(\Omega) \).

Therefore, the solution \( w = (u, p, \xi) \) is in \( Z = \mathcal{V} \times Q_0 \times L^2(\Omega) \).

**Remark 3.4**

- The domain of dissipation function \( D_{2m} \) is defined by

\[ \mathbb{D}^r = \{ (q, \eta) \in Q_0 \times L^2(\Omega) : |q|_{0,r} \leq \left( \frac{3}{2} C_m \eta \right)^{1/r} \quad \text{with} \quad \frac{1}{2m} + \frac{1}{r} = 1 \quad \text{and} \quad C_m = \frac{2^{(m-1)/2m}}{\sqrt{3}} \}. \]  

(3.36)

- Thus the inequality (3.30) makes sense if \( (\dot{p}, \dot{\xi}) \in \text{dom}(D_{2m}) = \mathbb{D}^r \), therefore the solution \( w = (u, p, \xi) \) will be sought in

\[ Z_r = \{ w = (u, p, \xi) \in Z : |q|_{0,r} \leq \left( \frac{3}{2} C_m \eta \right)^{1/r} \quad \text{with} \quad \frac{1}{2m} + \frac{1}{r} = 1 \quad \text{and} \quad C_m = \frac{2^{(m-1)/2m}}{\sqrt{3}} \}, \]  

(3.37)

which is a nonempty, closed and convex cone. Where, \( |x|_{0,r} = \left( \sum_{i=1}^{3} |x_i - x_{\text{mod}(i+1,3)}|^r \right)^{1/r} \) is a norm on \( \Omega \subset \mathbb{R}^3 \) and \( r = \frac{2m}{2m-1} \).
3.3.3 Variational formulation

According to lemma 3.3, we express the constitutive law by the form given in (3.30)

\[(\dot{p}, \dot{\xi}) \in \mathbb{D}^r, \sigma : (q - \dot{p}) + \chi : (\eta - \dot{\xi}) \leq D_{2m}(q, \eta) - D_{2m}(\dot{p}, \dot{\xi}), \quad \forall (q, \eta) \in Q_0 \times L^2(\Omega). \tag{3.38}\]

Let us multiply (3.32) by \(v - \dot{u}\) and integrate by parts over \(\Omega\), and use (3.33), therefore the variational formulation of (3.32) becomes

\[
\int_{\Omega} \sigma : (\varepsilon(v) - \varepsilon(\dot{u})) \, dx = \int_{\Omega} f(v - \dot{u}) \, dx + \int_{\Gamma_N} g(v - \dot{u}) \, ds, \quad \forall v \in \mathcal{V}. \tag{3.39}\]

The integral form of (3.38) over \(\Omega\) is given by

\[(p, \xi) \in \mathbb{D}^r, \int_{\Omega} \sigma : (q - \dot{p}) \, dx + \int_{\Omega} \chi : (\eta - \dot{\xi}) \, dx \leq \int_{\Omega} D_{2m}(q, \eta) \, dx - \int_{\Omega} D_{2m}(\dot{p}, \dot{\xi}) \, dx, \quad \forall (q, \eta) \in Q_0 \times L^2(\Omega). \tag{3.40}\]

Subtract (3.40) from (3.39), and substitute (3.24) and (3.26) to obtain

\[
(p, \xi) \in \mathbb{D}^r, \int_{\Omega} \mathbb{C}(\varepsilon(u) - p) : [\varepsilon(v) - \varepsilon(\dot{u}) - (q - \dot{p})] \, dx + k \int_{\Omega} \xi : (\eta - \dot{\xi}) \, dx + \int_{\Omega} D_{2m}(q, \eta) \, dx
\[
- \int_{\Omega} D_{2m}(\dot{p}, \dot{\xi}) \, dx \geq \int_{\Omega} f(v - \dot{u}) \, dx + \int_{\Gamma_N} g(v - \dot{u}) \, ds, \quad \forall (v, q, \eta) \in Z. \tag{3.41}\]

From (3.41) we thus obtain the time-dependent variational inequality

\[
a(w(t), z - \dot{w}(t)) + j_{2m}(z) - j_{2m}(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle \quad \forall z \in Z, \tag{3.42}\]

where \(z = (v, q, \eta), a(., .)\) is a bilinear form, \(\ell(.)\) a linear functional and \(j(.)\) a nonlinear functional defined by

\[
a(., .) : Z \times Z \rightarrow \mathbb{R}, \quad a(w, z) = \int_{\Omega} \mathbb{C}(\varepsilon(u) - p) : (\varepsilon(v) - q) \, dx + k \int_{\Omega} \xi \eta \, dx, \tag{3.43}\]

\[
\ell(t) : Z \rightarrow \mathbb{R}, \quad \langle \ell(t), z \rangle = \int_{\Omega} f(t) \, v \, dx + \int_{\Gamma_N} g(t) \, v \, ds. \tag{3.44}\]

\[
j_{2m} : Z \rightarrow \mathbb{R}, \quad j_{2m}(z) = \int_{\Omega} D_{2m}(q, \eta) \, dx. \tag{3.45}\]

We assume

\[
f \in L^2(0, T; \mathcal{V}), \quad g \in L^2(0, T; (L^2(\Gamma))^3) \quad \text{and} \quad w(0) = 0. \tag{3.46}\]

The primal variational problem of elastoplasticity thus takes the following form.

For a given \(\ell \in H^1(0, T; \mathcal{V}'),\) with \(\ell(0) = 0,\) find \(w = (u, p, \xi) \in H^1(0, T; Z)\) with \(w(0) = 0,\) such
that for almost all \( t \in (0, T) \), \( \dot{w}(t) \in Z_r \) and
\[
a(w(t), z - \dot{w}(t)) + j_{2m}(z) - j_{2m}(\dot{w}(t)) \geq \langle \ell(t), z - \dot{w}(t) \rangle, \text{ for all } z \in Z_r. \tag{3.47}
\]

### 3.4 Existence and uniqueness of the solution of the primal variational problem (3.47)

In this section we focus on the case which \( m \) lies in \((1, +\infty)\). The particular case \( m = 1 \) and \( m = \infty \) are respectively well known as Von-Mises and Tresca criteria. Therefore the dissipation function is given by
\[
D_{2m}(q, \eta) = \begin{cases}
\sqrt{2} \frac{\sigma_0}{C_m} |q|_{0,r} & \text{if } |q|_{0,r} \leq \sqrt{2} C_m \eta \\
\infty & \text{if } |q|_{0,r} > \sqrt{2} C_m \eta, \text{ with } \frac{1}{2m} + \frac{1}{r} = 1 \text{ and } \eta \geq 0.
\end{cases} \tag{3.48}
\]
its domain is
\[
\mathbb{D}^r = \{(q, \eta) \in Q_0 \times L^2(\Omega) : |q|_{0,r} \leq \sqrt{3} C_m \eta\}
\]
and
\[
Z_r = \{z = (v, q, \eta) \in Z : |q|_{0,r} \leq \sqrt{3} C_m \eta\}. \tag{3.49}
\]
which is a nonempty, closed and convex cone. In addition \( |x|_{0,r} = \left( \sum_{i=1}^{3} |x_i - x_{\text{mod}(i+1, 3)}|^r \right)^{1/r} \) is a norm on \( Q \subset \mathbb{R}^3 \) and \( C_m = \frac{\gamma^{(m-1)/2m}}{2^{(m+1)/3}} \), and \( 1 < r = \frac{2m}{2m-1} < 2 \). On the other hand, \( z = (v, q, \eta) \in Z_r \) imposes a constraint on the relation between the components \( q \) and \( \eta \). We introduce the assumption
\[
z = (v, q, \eta) \in Z_r \implies |q|_{0,r}^2 \leq 2^{-1/m} \eta^2. \tag{3.50}
\]

**Remark 3.5** Since \( Q \) is a finite dimensional subspace in \( \mathbb{R}^3 \), therefore the norm \( . |_{0,r} \) is equivalent to the Euclidean norm \( |x|^2 = \sum_{i=1}^{3} |x_i|^2 \) on \( \mathbb{R}^3 \), i.e there exist \( (\alpha_1, \alpha_2) \in \mathbb{R}^*_+ \) such that
\[
\alpha_1 |p| \leq |p|_{0,r} \leq \alpha_2 |p|, \quad \forall p \in Q. \tag{3.51}
\]

**Theorem 3.1** Under the assumptions (3.35),(3.46),(3.48),(3.49), (3.50) and the results(3.51), there exists a unique solution \( w \in H^1(0, T; Z) \) for the primal variational problem (3.47) for \( m \in (1, +\infty) \). In addition this solution depends Lipschitz continuously on the parameters of the problem.
Proof

From (3.25) and (3.43) it is obvious that \( a(.,.) \) is symmetric i.e
\[
a(z_1, z_2) = a(z_2, z_1) \quad \forall z_i = (v_i, q_i, \eta_i) \in Z, \quad i = 1, 2.
\]

On the other hand, \( a(.,.) \) is \( Z \)-elliptic on \( Z_r \).
\[
a(z, z) = \int_{\Omega} [\mathcal{C}(\varepsilon(v) - q) : (\varepsilon(v) - q) + k\eta^2] \, dx \quad \forall z = (v, q, \eta) \in Z_r.
\]

Indeed, using (3.25), (3.50), (3.51) and from Young’s inequality, we have for \( z = (v, q, \eta) \in Z_r \)
\[
\mathcal{C}(\varepsilon(v) - q) : (\varepsilon(v) - q) + k\eta^2 = 2\mu(\varepsilon(v) - q) : (\varepsilon(v) - q) + \lambda(tr(\varepsilon(v) - q))^2 + k\eta^2,
\]
where \( \mu > 0, \lambda > 0, \)
\[
\geq 2\mu|\varepsilon(v) - q|^2 + \frac{1}{2}k\eta^2 + \frac{1}{2}k\eta^2,
\]
\[
\geq 2\mu(|\varepsilon(v)|^2 - 2\varepsilon(v) : q + |q|^2) + \frac{1}{2}k^{1/m}|q|_{0,r}^2 + \frac{1}{2}k\eta^2,
\]
\[
\geq 2\mu(|\varepsilon(v)|^2 - d|\varepsilon(v)|^2 - d^{-1}|q|^2 + |q|^2) + \frac{1}{2}k^{1/m}|q|_{0,r}^2 + \frac{1}{2}k\eta^2, \quad 0 < d < 1,
\]
\[
\geq 2\mu(1-d)|\varepsilon(v)|^2 + |2\mu(1-d^{-1}) + \frac{1}{2}k(\alpha_1)^22^{1/m}|q|^2 + \frac{1}{2}k\eta^2, \quad 0 < d < 1.
\]

Choosing \( d \in (\frac{2\mu}{k(\alpha_1)^22^{1/m} + 2\mu}, 1) \), we obtain
\[
\mathcal{C}(\varepsilon(v) - q) : (\varepsilon(v) - q) + k\eta^2 \geq c_0(|\varepsilon(v)|^2 + |q|^2 + \eta^2), \quad \forall z = (v, q, \eta) \in Z_r,
\]
using the Korn’s inequality we obtain,
\[
a(z, z) \geq c_0||z||_{Z_r}^2, \quad \forall z = (v, q, \eta) \in Z_r.
\]

Therefore, \( a(.,.) \) is \( Z \)-elliptic on \( Z_r \), where \( r = \frac{2m}{2m-1} \) and \( m \in (1, +\infty) \).

**Boundedness of the bilinear for \( a(.,.) \)**

\[
|a(w, z)| \leq \int_{\Omega} (2\mu|[\varepsilon(u) - p] : (\varepsilon(v) - q)] + \lambda|tr(\varepsilon(u))||tr(\varepsilon(v))|) \, dx + k \int_{\Omega} |\xi||\eta| \, dx \quad \forall w, z \in Z,
\]
\[
\leq 2\mu|\varepsilon(u) - p||L^2(\Omega)||\varepsilon(v) - q||L^2(\Omega) + \lambda||\varepsilon(u)||L^2(\Omega)||\varepsilon(v)||L^2(\Omega) + k||\xi||L^2(\Omega)||\eta||L^2(\Omega),
\]
\[
\leq 4\mu||w||_{Z}||z||_{Z} + \lambda||w||_{Z}||z||_{Z} + k||w||_{Z}||z||_{Z},
\]
\[
\leq c||w||_{Z}||z||_{Z} \quad \forall w, z \in Z,
\]
where $a(\cdot, \cdot)$ is continuous and bounded. On the other hand, $j_{2m}$ is nonnegative, convex and positively homogeneous. This follows from the properties of $D_{2m}$. In addition $\forall w, z \in Z$

$$|j_{2m}(w) - j_{2m}(z)| = |\int_{\Omega} \sqrt{\frac{3}{2}} \sigma_0 (p|0,r - |q|0,r) dx|,$$

where $1 - 2m = \frac{1}{2m} + \frac{1}{r} = 1$,

$$\leq \sqrt{\frac{2}{C_m}} \int_{\Omega} |p - q|0,r dx,$$

$$\leq 2^{1/2m}(\text{meas}(\Omega))^{1/2} \|p - q\|_{L^2(\Omega)} \quad (\text{Cauchy-Schwartz inequality}),$$

$$\leq \alpha_2 \sigma_0 2^{1/2m}(\text{meas}(\Omega))^{1/2} \|p - q\|_Z,$$

where $j_{2m}(\cdot)$ is Lipschitz continuous on $Z_r$. Thus, by applying theorem 7.5 in [[8], page 161] the primal problem (3.47) has a unique solution for $m \in (1, +\infty)$ and it depends continuously on the properties of the material.
Conclusion and perspectives

The increasing complexity of the problems which have to be solved by analysts and engineers, as well as the abundant experimental data that are available, have stimulated the development of advanced constitutive models. The rate-independent behavior of Geo-materials can be described within the framework of elastoplasticity. In this theory, the strain tensor is decomposed additively into a reversible part which describes elastic behavior, whereas the irreversible characterizes the plastic feature. In the classical model, followed throughout this work, stresses are allowed to lie in yield domain which is assumed to be convex and closed.

The purpose of this project was to explore some theoretical aspects of the classical primal problem of elastoplasticity as settled by Han and Reddy in [8]. Our goal here was to present the mathematical models, of the evolution of an elastoplasticity body in the case of quasi-static, infinitesimal deformation, with both kinematic and isotopic hardening. Besides that, we prove both the existence and uniqueness of the solution of the primal problem.

The major outcome of this essay lies in chapter 3, where from the yield function \( f \) defined in section (3.1) we have constructed and established the well-posedness of the dissipation function for \( 1 \leq m \leq \infty \). In addition, we have applied this result to prove both existence and uniqueness of the solution of primal problem in the case of \( 1 < m < \infty \).

Our future perspectives is firstly, the computational implementation of the model in order to simulate some results presented in this work. Secondly, we will try to improve those results or extend them to the case of gradient plasticity. We would like also to extend our research in the framework of viscoplastic model that can provide better results, and can approach the material behavior in a more realistic way (not quasi-static). In fact, in many cases and with certain materials, elasto-viscoplastic models are the only way to reproduce the experimental measurements.
Appendix

Figure 3.1: The Haigh-Westergaard stress space.

Proof Lemma 3.1
Let assume that $|\sigma|_{0,r} = |p|_{0,s} = 1$, for $1 \leq r \leq \infty$,

- for $1 < r < \infty$, we have
\[
\sigma : p = \sum_{i=1}^{3} [\sigma - \sigma_{\text{mod}(i+1,3)}]([p]_i - [p]_{\text{mod}(i+1,3)}),
\]
\[
\leq \sum_{i=1}^{3} ||\sigma - \sigma_{\text{mod}(i+1,3)}||[p]_i - [p]_{\text{mod}(i+1,3)}|,\]
\[
\leq \frac{1}{r} \sum_{i=1}^{3} ||\sigma - \sigma_{\text{mod}(i+1,3)}||^r + \frac{1}{s} \sum_{i=1}^{3} ||p]_i - [p]_{\text{mod}(i+1,3)}|^s \quad \text{(Young's inequality)},
\]
\[
= \frac{1}{r} + \frac{1}{s},
\]
\[
= 1,
\]
therefore \( \sigma : p \leq 1 \) for \( |\sigma|_{0,r} = |p|_{0,s} = 1 \).

On the other hand, \( |\sigma : p| \leq |\sigma|_{0,r}|p|_{0,s} \forall \sigma \in \mathbb{Q}', p \in \mathbb{Q} \) and \( \frac{1}{r} + \frac{1}{s} = 1 \).

- For \( m = \infty \),
\[
\sigma : p = \sum_{i=1}^{3} [\sigma - \sigma_{\text{mod}(i+1,3)}]([p]_i - [p]_{\text{mod}(i+1,3)}),
\]
\[
\leq \sum_{i=1}^{3} ||\sigma - \sigma_{\text{mod}(i+1,3)}||[p]_i - [p]_{\text{mod}(i+1,3)}|,\]
\[
\leq \max_{i,j \in \{1,2,3\}} ||\sigma - \sigma_j|| \sum_{i=1}^{3} ||p]_i - [p]_{\text{mod}(i+1,3)}|^r
\]
\[
= |\sigma|_{0,\infty}|p|_{0,1}.
\]
Thus, \( \sigma : p \leq |\sigma|_{0,\infty}|p|_{0,1} \).

**Proof Proposition 3.3**

\[
\forall (q, \mu) \in \mathbb{Q} \times \mathbb{R}^+ \quad D_{2m}(q, \mu) = \sup_{(\sigma, \chi) \in \mathcal{K}_{2m}} (\sigma : q + \chi \mu),
\]
\[
\leq \sup_{|\sigma|_{0,2m} \leq \mathcal{C}_m^{\frac{1}{2m}}(\sigma_0 - \chi)} |\sigma|_{0,2m}|q|_{0,r} + \chi \mu,
\]
with \( \frac{1}{2m} + \frac{1}{r} = 1 \) (lemma 3.1),
\[
\leq \sup_{\chi \leq 0} \left( \frac{\sqrt{2}}{\mathcal{C}_m} \sigma_0 |q|_{0,r} + \left( \mu - \frac{\sqrt{2}}{\mathcal{C}_m} |q|_{0,r} \right) \chi \right),
\]
with \( \frac{1}{2m} + \frac{1}{r} = 1 \),
\[
\leq \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{\mathcal{C}_m} \sigma_0 |q|_{0,r} & \text{if } |q|_{0,r} \leq \frac{\sqrt{2}}{\mathcal{C}_m} \mu \\
\infty & \text{if } |q|_{0,r} > \frac{\sqrt{2}}{\mathcal{C}_m} \mu
\end{array} \right. \]
with $\frac{1}{2m} + \frac{1}{r} = 1$.

On the other hand, for

$$\sigma = \frac{\sqrt{\frac{3}{4}}}{C_m |q|_{0,r}} \sigma_0 \left( |q_1 - q_2|^r (q_1 - q_2), |q_2 - q_3|^r (q_2 - q_3), |q_3 - q_1|^r (q_3 - q_1) \right),$$

with $\frac{1}{2m} + \frac{1}{r} = 1$ and $\chi = 0$,

therefore, $\Sigma = (\sigma, 0) \in \mathbb{K}_{2m}$, and in addition,

$$\sigma : q = \frac{\sqrt{\frac{3}{4}}}{C_m} \sigma_0 |q|_{0,r} \leq \sup_{(\tau, \gamma) \in \mathbb{K}_{2m}} (\tau : q + \gamma \mu) = D_{2m}(q, \mu).$$

Thus, the dissipation function is given by

$$D_{2m}(q, \mu) = \begin{cases} 
\frac{\sqrt{\frac{3}{4}}}{C_m} \sigma_0 |q|_{0,r} & \text{if } |q|_{0,r} \leq \sqrt{\frac{3}{2}} C_m \mu \\
\infty & \text{if } |q|_{0,r} > \sqrt{\frac{3}{2}} C_m \mu,
\end{cases}$$

with $\frac{1}{2m} + \frac{1}{r} = 1$.

and $C_m = \frac{2^{(m-1)/2m}}{\sqrt{3}}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{yield_surface.png}
\caption{Trace on the deviatoric plane of the Von Mises-Tresca type yield surface.}
\end{figure}
Bibliography


